Estimated FFT Interval

Elisa A. M. Arcanjo [a1], Erivelton G. Nepomuceno [b] and Leonardo B. Felix [c]
[a] [b]Departamento de Engenharia Elétrica, Universidade Federal de São João del-Rei, São João del-Rei, MG, Brazil
[c]Núcleo Interdisciplinar de Análise de Sinais, Universidade Federal de Viçosa, Viçosa, MG, Brazil

Abstract

This paper presents a study of interval arithmetic concepts applied in the Fast Fourier Transform (FFT) calculations. The main objetive is to show the errors arising from equipment and measurements taken into account when FFT is calculated. The errors serves to define the lower and upper limits of the signal in the time domain. The FFT is calculated taking the values within these limits. It is shown that the average FFT signal variance has an asymptotic behavior in relation to the number of events held. The final standard deviation used for frequency spectrum is the final asymptotic average.

Keywords: Fast Fourier Transform, FFT, Interval Arithmetic, Fourier Transform, Errors.

1. Introduction

The Fast Fourier Transform (FFT) is an efficient algorithm for computing the Discrete Fourier Transform (DFT). The FFT is an essential element for the areas of math, science, engineering, signal processing, mechanical, electrical, chemical, physical, medical areas and telecommunications. It is also a powerful tool in the analysis of linear systems, because there are cases in which analysis in the frequency domain is easier than in the time domain [1, 2, 3, 4].

There are studies that suggested a significant presence of erros in FFT calculations, including rounding and truncation errors [1, 5, 6, 7, 8, 9, 10]. Searchs that consider these errors generated in differents calculations gained strength and scientific foundation to the implementation of floating-point arithmetic as described in IEEE Standard [11]. And, new searchs showed to that the errors from rounding and truncation, for certain functions and initial conditions, are significant and incompatibles with the expected mathematically [12, 13, 14, 15, 16].

[1]E-mail Corresponding Author: elisaarcanjo@hotmail.com

Although those studies are important, the pre-defined error rates in the technical specifications of a tool and the interferences during the data collection, commonly, not are taken into account. This gap can generate inconsistent results with actual process and, even using supercomputers, you need to consider and implement these types of errors before any estimate or computational analysis. An alternative approach to resolve the problem was demonstrated by Liu and Kreinovich (2010) where they have proposed, in a first analysis, an FFT algorithm considering a fixed error and time-invariant for the entire signal range. The steps of the FFT are replaced by corresponding operations on interval arithmetic and they demonstrate how to do the calculations of the convolution and the FFT, in order to get the intervals represented of the output signal $y(t)$ at time $n.log(n)$. In their studies, rounding and truncation errors are not considered because, in signal processing applications, these errors are generally insignificant compared to the measurements errors [4].

Therefore, the purpose of this article is to describe how to find the spectrum limits from the use of existing functions, statistical concepts (mean and standard deviation) and the concept of Infinite Evaluation of Arithmetic Interval. Using these concepts will not be necessary to rebuild the FFT algorithm to find the tracks and signal limitations in the frequency domain.

2. Preliminary Concepts

In recent decades, a large part of the systems based on continuous time analog circuits are now implemented through digital discrete-time systems (A/D). This phenomenon is due, in large part, to the ease of access to dedicated cards and general purpose computers. Thus, scientists and engineers seek to represent the physical phenomena in functions of continuous variables and differential equations. Different numerical techniques are used and developed to solve these equations and functions [17].

The A/D conversion, functions and equations that represent a real physical phenomenon can contain errors. There are three kind of errors in numerical computation: error propagation in the data and initial parameters, rounding error and the truncation error. Rounding and truncation errors, as seen in [4], in signal processing applications are insignificant compared to the measurement errors and parameter data. Then, to calculate an FFT, priority should take into account the errors of data collection, equipment and initial settings.

2.1. Fourier Transform

For Fourier Transform calculations is necessary to have the sampled values of $x(t)$ because a digital computer requires discrete values. This to say that the computer needs to sequence of numbers to do their jobs. Thus, when using the samples of $x(t)$ a limited signal in time and relate to samples $X(\omega)$ (values in the frequency domain) to obtain spectral sampling signal. This sample has the following characteristics [3]:

- f_0 (initial frequency);
- $\frac{1}{T_0}$ (time of sampling);

A sampled signal is repeated periodically every T seconds and the sampled spectrum also repeats periodically [3].

The formula that defines DFT according [3] is:

$$X_r = \sum_{n=0}^{n_o-1} x_n e^{-jr\Omega_0 n} \tag{1}$$

and $\Omega_0 = \frac{2\pi}{N_0}$, $r = 0, 1, 2, ..., N_0 - 1$ and N_0 = sample period.

DFT or FFT is required the initial period of sampling N_0, the sampling time T_0 and sampling frequency f, which must be, at least, twice the highest frequency in Hz of sign. It is important to emphasize that even the FFT reducing the number of calculations, it will generate the same answers that the DFT [3].

According to Lathi(2007), Fast Fourier Transform (FFT) is an algorithm which reduces the calculation time n^2 steps until $n.log_2(n)$. The only requirement is that the number of values in the series is a power of 2 [3].

2.2. Interval Arithmetic

Consist of an alternative to reach limits that can guarantee the result within the values range of the lower and upper limits, through strict and automatic control of the error of the result. The intervals analysis aims to answer the question of accuracy and efficiency in the practice of Scientific Computing. It is interested in techniques that can be programmed by computer, containing in its computing rigorous analysis, complete and automatic of errors of result [18, 19, 20].

In this way, interval algorithms, in contrast to the specific algorithms, compute a range as a solution, with the assurance that the answer belongs to this range [15]. So, interval results always carry the security of their quality and the degree of their uncertainty, because the diameter of a interval solution is indicative of input data error error influence on the final result. This is a type of sensitivity analysis, which can replace executions of repeated simulation and expensive [18] .

The operations sum, subtraction, multiplication and division are described below [20].

1. Interval Adding: $[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$;
2. Interval Subtraction: $[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$;
3. Interval Multiplication: $[x] \times [y] = [mim\{\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y}\}, max\{\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y}\}]$;
4. Interval Division: $\frac{[x]}{[y]} = [x] \times \frac{1}{[y]}$

3. Methodology

These proposed study disregard rounding and truncation errors, since such errors are insignificant compared to the measurement errors, how is described in Liu and Kreinovich (2010).

The simulated signal for this article was constructed from a known theoretical signal function, described below.

$$x(t) = X_0 + X_1 \cos(2\pi f_1 t + \frac{\pi}{2}) + X_2 \cos(2\pi f_2 t + \frac{\pi}{6}) \tag{2}$$

$X_0 = X_1 = X_2 = 20;\ f_1 = 50; f_2 = 100$ and sampling frequency $F_s = 8(f_2)$;

The signal representation in the time domain can be seen in Figure 1.

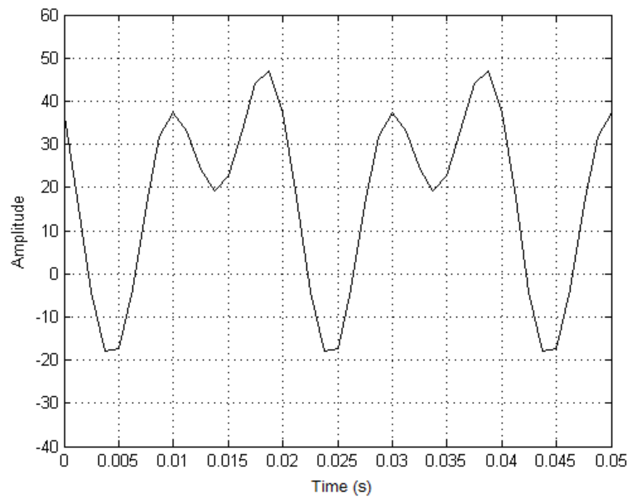

Figure 1 - Signal in the time domain.

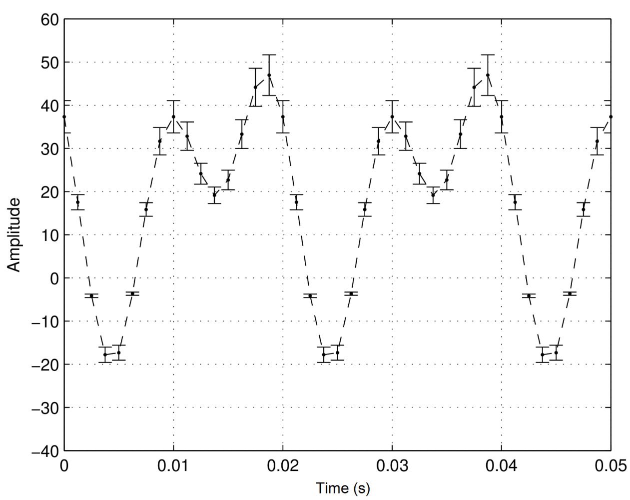

Figure 2 - Signal and tolerance of 10%

To define the sign limits was considered a hypothetical error of 10%. The choice of this value is based on the percentage of errors found in capacitors, resistors and other electrical equipment. However, this error can vary over time, but for this first time consider a fixed error during the entire sign. Then, applying the error to the input find the lower limit and the upper limit, also called signal intervals. Figure 2 shows the original signal and the limits. The estimated range for the FFT signal was based on the Finite Rating theorem according described in Moore(1979). The definition is described below.

Let M_1 and M_2 arbitrary set and let $g : M_1 \longrightarrow M_2$ arbitrary mapping (function) of M_1 in M_2. Denote by $S(M_1)$ and $S(M_2)$ families of subsets of M_1 and M_2 respectively. According W. Strother [21], is called the set-valued mapping, $\overline{g} : S(M_1) \longrightarrow S(M_2)$,

$$\overline{g}(X) = \{g(x) : x \in X, X \in S(M_1)\} \tag{3}$$

the extension unit g. You can also write

$$\overline{g}(X) = \cup_{x \in X} \{g(x)\}. \tag{4}$$

Therefore, $\overline{g}(X)$ is the union M_2 of all sets contained in a single element $g(x)$ for some x in X. Sometimes $\overline{g}(X)$ is simply referred to as "the image of the mapping g the set X".

But even if g and X have a finite representation, as a range, in general, still not have the finite representation to $\overline{g}$. Therefore, estimates are required of $\overline{g}(X)$ considering different quantities and values in X. This estimate was made as follows:

1. Each point of the original signal is represented by an interval $X = [a, b]$ and the limits a and b of interval are defined according to the error 10% described above.

2. To select a value within the range X, the random function for ranges was implemented in the algorithm. The random function is performed within a repetition block ($for()$) according to the size of the original signal.

3. Each new value found within the intervals given rise to a new representation of the original signal.

4. This new representation of the signal is passed on to the function $fftn()$. This function makes the calculations of the Fast Fourier Transform.

5. The results generated by this function are saved in array A with dimensions L x C, where L is the amount of times the FFT is calculated and C is the size of signal. The signal size is always the same in all cases.

6. First, it is calculated 5 FFTs considering the random values selected within the interval X. Then, the same steps were repeated to calculate 10, 15, 20 until $1,000$ FFTs, of 5 in 5. This process occurred 200 times.

7. Using the results of the matrix A, is calculated the standard deviation of FFTs with the function $std()$. After, the mean of standard deviation is calculated using the function $mean()$.

8. And, it is estimated the average of the averages to define the variation of the spectrum. This variation is added and subtracted from the result of the FFT of the signal and the upper and lower limits of the spectrum are defined.

4. Results

The main result obtained was the average standard deviation and its graphical representation can be seen in Figure 3. It is can see, in the axis x, the amounts of FFTs and, the axis y, average of standard deviations each set. It is observed in this graphic an asymptotic behavior.

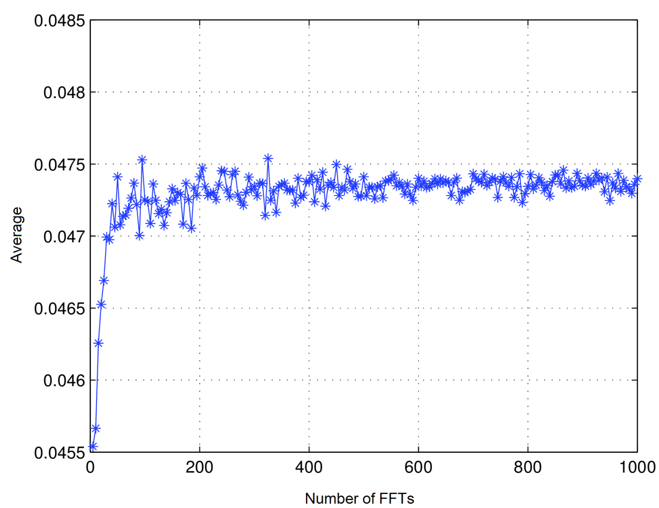

Figure 3 - Average standard deviation of calculated FFTs Sets

Therefore, the final standard deviation the considered for Fourier spectrum is the average final asymptotic value 0.0472 (multiplied by three to guarantee the results). Figure 4 shows signal spectrum appling the final standard deviation and shows the boundaries in the frequency domain.

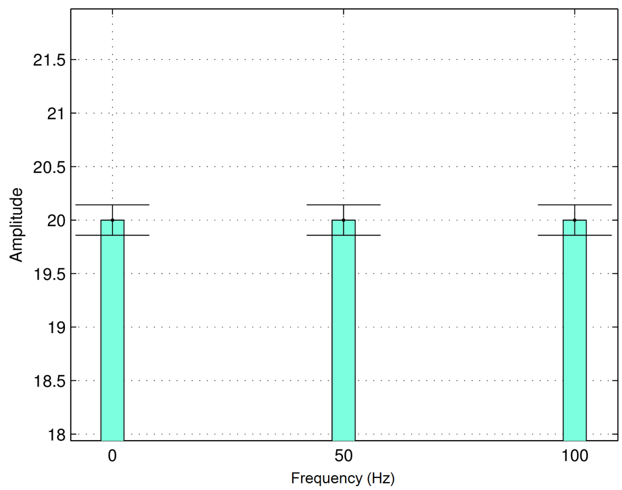

Figure 4 - Signal in the Frequency Domain applying the final standard deviation (3×).

5. Conclusion

According researches associated to Fast Fourier Transform, one must take into consideration, before any analysis or calculation, problems generated during data collection and error percentage of equipment and electrical signals generators. Such errors can influence in the final results of each analysis.

To address and resolve the problem found, resorted the concepts of interval arithmetic. Concepts that define the limits and intervals for a signal before and after calculations.

So, this article demonstrates that, when such errors are considered and applied to the signal in the time domain, before the calculation of the FFT, in the frequency domain is also necessary to consider the interference of this error in the resulting spectrum. The final results show that it is possible to define the maximum and minimum of the significant frequencies of the signal in the Fourier spectrum.

Recalling that all tests were done considering an initial fixed error. However, it is necessary to make analysis for errors variants in time, because in fact most real situations of daily vary over time.

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