INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

NUMERICAL IMPRECISION AND ITS IMPACT ON DISCRETE SYSTEMS SUCH AS LOGISTIC MAP

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Abstract: A well-known property of chaotic systems is the sensitivity to initial conditions that are typically observed by means of numerical methods. However, how stay the situation, in that the tool we use to view the chaotic behavior interferes in results due to their inability in accurately represent the initial condition. Soon a system with chaotic behavior that is subject to these small interperes can present a totally unexpected result. But as can be seen the change in the rounding mode appears as an ally in combating these misleading results.

keywords: Discrete Dynamical Systems, Chaos, Numerical Simulation.

1. INTRODUCTION

The field of nonlinear dynamic systems has attracted the attention of researchers around the world [**?**]. What a long time are looking for ways to understand the behavior of chaotic nonlinear systems, independent of dynamics even be continuous or discrete. It is noted that the nonlinear discrete equations or recursive functions have a great importance in many processes. As stated in [**?**] that recursive functions, described as the following equations

$$x_{n+1} = f(x_n), \tag{1}$$

are used to describe a variety of problems.

Examples of important recursive functions have been one-dimensional maps, especially the logistic map, which according to [**?**] is probably one of the most used in various applications.

The analysis is based on logistic equation to study the stability of their fixed points, which maybe stable or unstable. However to occur this analysis is necessary first obtained the bifurcation diagram. We did not find in the literature a strict procedure for their numerical building of bifurcation diagram. The procedure suggested in [**?**] does not take into account numerical problems due to rouding off. And this is also stated in [**?**], in which a hypothesis about the possible interference of the computer in the chaos of study is raised.

This work presents a way to reduce the numerical limitations by varying the rounding mode.

2. PURPOSE

This paper presents the implications of numerical limitations for the simulation of discrete systems, such as the logistic map. In addition, it presents a chance to overcome the numerical errors by rounding mode change when this technique works together with theorem proposed by [**?**] for convergence of recursive functions on computers.

3. METHODS

Consider discrete logistic model [**?**]. Let the interval $I = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$, $f : I \to \mathbb{R}$ a continuous function and $\varepsilon > 0$. Consider f in the following form

$$x_{n+1} = f(x_n) = rx_n(1 - x_n), \tag{2}$$

where $n \in \mathbb{N}$ and $r \in \mathbb{R}$ and $r > 1$

Let $A \subset I$ be defined as

$$A = \{a \in I \ | \ a = 1/r\}. \tag{3}$$

It is easy to show that when $x_0 = a$

$$x_1 = f(a) = r\frac{1}{r}\left(1 - \frac{1}{r}\right) = \frac{r-1}{r}, \tag{4}$$

and $x_1 = x_2 = \ldots = x_n$ and so $d(f^p(a), x^*) \to 0$ as $p \to \infty$. In particular, $d(f^p(a), x^*) = 0$ when $p = 1$. It is known that $x^* = (r-1)/r$ is a fixed point of (2).

Let the definition of fixed point proposed in [**?**]

Definition 1: If $d(\widehat{f}_n(\widehat{x}^*), \widehat{f}_{n-1}(\widehat{x}^*)) > \delta_n + \delta_m$, then $\widehat{x}^*$ is a fixed point.

where $\widehat{f}_n$ and $\widehat{f}_{n-1}$ be an approximation of f_n and f_{n-1} respectively., $\widehat{x}^*$ approximation of x^*, δ_n and δ_m error obtained by iterating n and m.

So when the distance between two consecutive points is low than the accumulated error, the simulation must be interrupted.

In [**?**] was presented the situation where analytically convergence occurs on the first iteration, but numerically in some cases the convergence does not occur.Aware of this situation, is proposed to analyze the influence of rounding and thus verify that by varying the rounding technique can obtain the response obtained analytically

We used the following Matlab functions to analyze the influence of rounding:

- Used to round to the nearest number below:

  ```
  system_dependent('setround',-Inf)
  ```

- Used to round to the nearest number higher;

  ```
  system_dependent('setround',+Inf)
  ```

- Used to round to the nearest number;

  ```
  system_dependent('setround',0.5)
  ```

The default rounding method for the majority of numerical simulation software is the closest.

4. RESULTS

To evaluate the potential of the proposed methodology was defined as logistic map with control parameter $r = 4$ and initial condition $x_0 = 1/r$. Therefore analytical analysis we have the following result.

$$x_1 = f(1/4) = 4 \cdot \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{4-1}{4} = \frac{3}{4} \tag{5}$$

$$x_2 = f(3/4) = 4 \cdot \frac{3}{4}\left(1 - \frac{3}{4}\right) = \frac{4-1}{4} = \frac{3}{4} \tag{6}$$

and $x_1 = x_2 = \ldots = x_n$ and so $d(f^p(1/4, x^*) \to 0$ as $p \to \infty$. In particular, $d(f^p(1/4), x^*) = 0$ when $p = 1$. It is known that $x^* = 3/4$ is a fixed point of (**??**).

In Algorithm 1 (see Tab. 1) the used commands to obtain the bifurcation diagram of the logistic map are presented

Algoritmo 1
$r \leftarrow 4;$
$x_0 \leftarrow 1/4;$
$N \leftarrow 1500;$
for $n \leftarrow 0$ **to** N **do**
$\mid x_{n+1} \leftarrow r \cdot x_n(1 - x_n);$
end

Table 1 – Algorithm for simulation of logistic map.

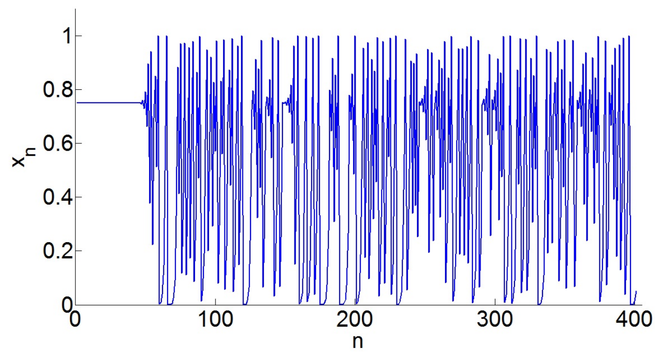

Figure 1 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$ **for** $1 \le r \le 4$**.**

when no specific rounding mode is preset, then the mode is the default Matlab.

The Algorithm 1 presented in Tab. **??** follows the methodology proposed by [**?**] and as a result provides Fig. **??**.

From what has been shown in Fig. **??** and also by the Lyapunov exponent of this time series can be concluded that it is a chaotic regime. However by means of the analytical results it is possible to reach a different conclusion.

Now they will present the results obtained when the rounding method varies between three possible options, rounded to $+\infty$; $-\infty$ and nearest value. The algorithm used for this step have the structure as default may be given in Algorithm **??** (see Tab. 2).

Algoritmo 2
$r \leftarrow 4;$
$x_0 \leftarrow 1/4;$
$N \leftarrow 1500;$
for $n \leftarrow 0$ **to** N **do**
$\mid x_{n+1} \leftarrow round^+(r \cdot x_n(1 - x_n));$
end

Table 2 – Algorithm for simulation of logistic map using rounding techniques.

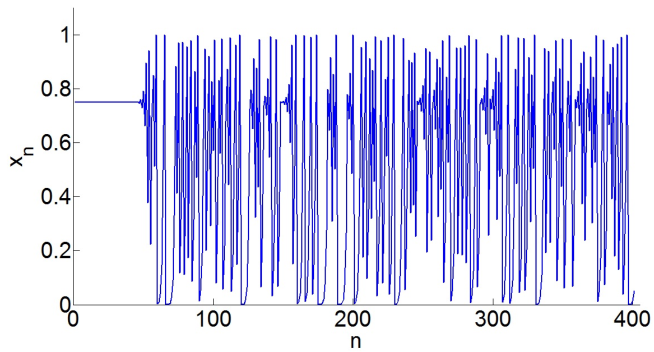

Figure 2 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$, when rounding technique is to $+\infty$.

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000002
2	0.750000000000000	3fe7ffffffffffe
3	0.750000000000001	3fe8000000000006
4	0.749999999999999	3fe7fffffffffff6

Table 3 – Numeric data for rounding to $+\infty$.

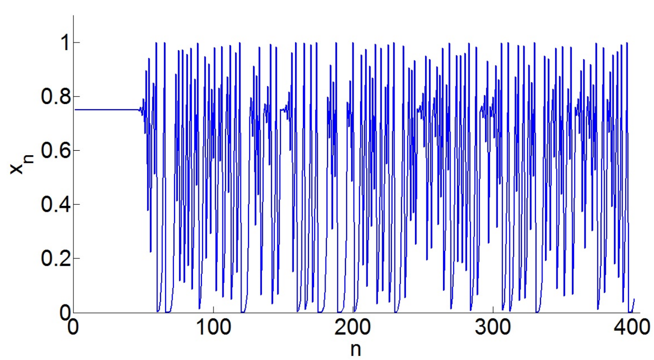

Figure 3 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$, when rounding technique is the closet.

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000002
2	0.750000000000000	3fe7ffffffffffe
3	0.750000000000001	3fe8000000000006
4	0.749999999999999	3fe7fffffffffff6

Table 4 – Numeric data for rounding to the closet.

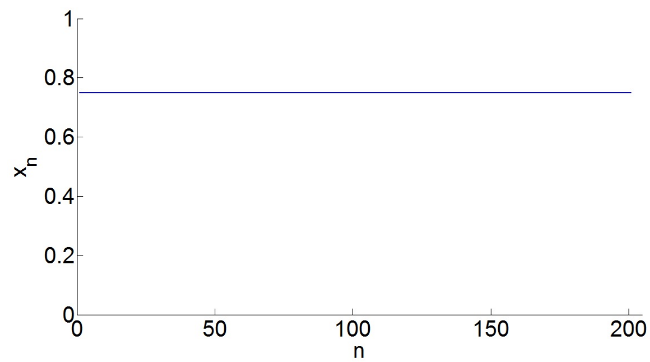

Figure 4 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$, when rounding technique is to $-\infty$, with 200 iterations.

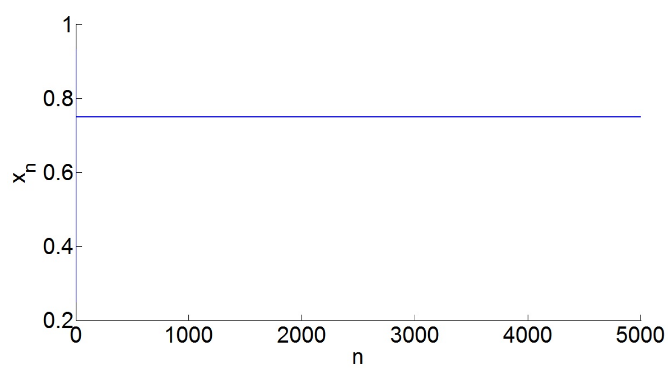

Figure 5 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$, when rounding technique is to $-\infty$, with 5000 iterations

5. DISCUSSION

After the simulations can be seen that as expected the results when simulating the logistic map without a predefined rounding method is the same simulation result using the rounding technique for the closest, because as can be found in the literature this option is the most common default option.

Moreover, it can be seen that the simulation using only rounding to $-\infty$ has a numerical result which is consistent with the analytical result.

The results obtained using the other two rounding techniques, initially present convergence for fixed point, however the errors inserted by round-off imply after few iterations divergence of fixed point, because in function of control parameter, the fixed point is in a bowl of chaotic attraction where small perturbations make a big difference due to high dependence on the initial conditions.

However if the theorem proposed by [**?**] for simulation of recursive functions on computers is applied it's possible calculate the error at each iteration, for example, the calculation error would x_1 equals $3.053113317719181 \cdot 10^{-16}$ and each iteration of this error increases because it is cumulative.

However if disregarded the increase in error after the calculation of the first iteration, calculating the distance between $x_0 = 0.250000000000000$ and $x_1 = 0.750000000000000$ is easy to observe that the cumulative error is much less than this value, so the simulation should continue, however ob-

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000000
2	0.750000000000000	3fe8000000000000
3	0.750000000000000	3fe8000000000000
4	0.750000000000000	3fe8000000000000

Table 5 – Numeric data for rounding to $-\infty$.

serving the distance between x_2 and x_1 we would have a value equal zero when observing the values given in decimal form (see Tab. **??** and Tab. **??**).

To ensure that the simulation should be stopped is observed the hexadecimal value. From Tab. **??** and **??** you can see that the values are different. However, this difference is less than the ulp (unit in last place) which equals $22.2204460492503131 \cdot 10^{-16}$ for the standard 64-bit double. One way to confirm this hypothesis is that if the difference was greater than that ulp in module the value in decimal would be equal to 0.750000000000001 or 0.749999999999999.

Soon the accumulated error is greater than the distance between x_2 and x_1, remembering that the errors arising from the calculation of x_1 and x_2 were disregarded, we then have that the most correct is to stop the simulation and take the value obtained in the third iteration as answer.

6. CONCLUSION

This short note presents the relevance of taking into account the method of rounding. The majority of software use the round mode to the nearest. It is clear that it is the best solution when one noticing only one arithmetic operation or when it is used to store a real number. However, this is not the case when the most important thing in the computation is a function. There is no easy to way to establish a correct rounding mode to a function, which in our opinion deservers further research.

ACKNOWLEDGMENTS

Thanks to CAPES, CNPq, and Fapemig for financial support and the PPGEL (UFSJ/CEFET-MG) for the opportunity to develop the MSc of Mr. Paiva.

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