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REVISITING HAMMEL ET AL. (1987): DOES THE SHADOWING PROPERTY HOLD FOR MODERN COMPUTERS?

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Abstract: Although computational techniques have been extensively applied in nonlinear science, the reliability evaluation of numerical results does not grow in the same pace. Hammel et al. (Journal of Complexity, 1987, 3(2), 136–145) have proved a theorem that a pseudo-orbit of a logistic map is shadowed by a true orbit within a distance of 10^{-8} for 10^7 iterates. We checked this theorem in a modern computational platform, and on the contrary, we found the simulation of the logistic map with less than 10^2 iterates presents an error greater than 10^{-8} .

keywords: Modeling, Numerical Simulation and Optimization, Nonlinear Dynamics and Complex Systems, Chaos and Global Nonlinear Dynamics, Pseudo-Orbits, Nonhyperbolic Maps.

1. INTRODUCTION

Numerical computing and Nonlinear Science and Complexity go hand in hand and this has been a long-term relation. In the 60s, it had already been observed the use of computers dedicated to efforts in understanding climate phenomena [1]. As stated by Macau and Lambruschini [2], computational techniques are applied to different topics in nonlinear dynamics, such as Synchronization, Bifurcation and

Chaos, Complex Networks, Conservative Systems and Nonlinear Partial Differential Equations. Lozi [3] also agrees that computing provides an essential contribution to the analysis of nonlinear dynamical systems. Thus, beyond using the computer to study systems of such nature, it also becomes important verifying the confiability of its numerical results. Some recent works have explored this issue of checking computer results [4–6]. Although, it is much more expressive the amount of research simply supported by computers. Added to the theoretical and experimental branches of science, computing is seen as a third one [7].

Hammel et al. [8] were among the first to examine the relation between numerical experiments and the true dynamics of a system [9]. By means of the Cray X-MP, a computer costing multi-million dollars at that time, they have reported the shadowing property as valid for a considerable number of iterates when simulating the logistic map for the parameter $r = 3.8$ and the initial condition $x_0 = 0.4$. When this case is considered, according to Hammel et al. [8], a pseudo-orbit of the logistic map is shadowed by a true orbit within a distance of 10^{-8} for 10^7 iterates. Since the theorem is proved for specific conditions, "it can be raised an issue if the computer test constitutes a sufficient condition, and therefore the theorem is proved for all cases, or whether it is a necessary condition, the result may not be valid for all cases", as states Nepomuceno and Martins [6].

Many researchers have applied this result in studies con-

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cerning the dynamical systems theory since then. More than 100 Hammel et al. [8] citations have been analyzed through the Scopus online platform and it was observed that, in some cases, the shadowing theorem is seen as a property possible to be generalized for non-hyperbolic systems. Petrov and Pilyugin [10] mention the paper of Hammel et al. [8] as the one that proves the shadowing property for systems with non-hyperbolic behavior. Hammel et al. [8] presents the "first proof of the existence of a shadow for a two-dimensional non-hyperbolic system over a non-trivial length of time", say Hayes et al. [9]. In addition, Chaitin-Chatelin [11] also mentions Hammel et al. [8] when points out considerations about the reliability of numerical results. Chaitin-Chatelin [11] remarks that the shadowing property had been shown valid for some initial condition and parameter values of the logistic map, without mentioning the cases for which the property fails.

2. PURPOSE

The aim of this study is to verify whether the shadowing property, as enunciated by Hammel et al. [8], is also valid for today's computer systems that adopt the IEEE 754-2008 standard. The same case as the one from that occasion will be studied, namely, the logistic map with $r = 0.4$ and the initial condition and $x_0 = 3.8$.

3. PRELIMINARY CONCEPTS

Essential concepts for understanding following sections appear here. The enumerated definitions and the lower bound theorem presented in this section are given by Nepomuceno and Martins [6].

3.1. IEEE 754-2008 standard and the Interval Arithmetic

The computer works with a limited representation of real numbers, once they are represented by intervals. Therefore, when performing certain mathematical operations, it might often not be possible to store their exact result on the computer memory. The use of round modes, as defined by the IEEE 754-2008 [12], is required. Given the combination of applying rounding methods and the way how the computer stores numbers, that is, given the interval arithmetic, mathematical properties such as commutativity, associativity and distributivity, cannot be guaranteed for floating-point operations [7]. It can be said that the floating-point simulations represent an abstraction of the reality and results from reality may not be easily transferred to the computer [13].

3.2. Orbits and Pseudo-orbits

Consider the following definitions for orbits and pseudo-orbits.

Definition 1 *An orbit is a sequence of values of a map, represented by $x_n = [x_0, x_1, \dots, x_n]$.*

Definition 2 *Let $i \in \mathbb{N}$ represents a pseudo-orbit, which is defined by an initial condition, an interval extension of f ,*

some specific hardware, software and numerical precision standard. A pseudo-orbit is an approximation of an orbit and we represent as

$$\hat{x}_{i,n} = [\hat{x}_{i,0}, \hat{x}_{i,1}, \dots, \hat{x}_{i,n}]$$

such that,

$$|x_n - \hat{x}_{i,n}| \leq \delta_{i,n}, \quad (1)$$

where $\delta_{i,n} \in \mathbb{R}$ is the error and $\delta_{i,n} \geq 0$.

3.3. Shadowing

Considering the definitions of orbit and pseudo-orbit, Hammel et al. [8] enunciate a theorem for the logistic map [14],

$$x_{n+1} = f(x_n) = rx_n(1 - x_n), \quad (2)$$

$n \in \mathbb{N}$ and $a \in \mathbb{R}$.

Through the use of notations for orbit and pseudo-orbit different from those mentioned on the previous definitions, they state that a pseudo-orbit $\{p_n\}_{n=0}^N$ is shadowed by a true orbit $\{x_n\}_{n=0}^N$ within shadowing distance $\delta_x = 10^{-8}$, with $p_0 = 0.4$ and $r = 3.8$, for $N = 10^7$.

3.4. Lower bound error

Theorem 1 *Let two pseudo-orbits $\{\hat{x}_{a,n}\}$ and $\{\hat{x}_{b,n}\}$ derived from two interval extensions. Let $\delta_{\alpha,n} = \frac{|\hat{x}_{a,n} - \hat{x}_{b,n}|}{2}$ be the lower bound error of a map $f(x)$, then $\delta_{a,n} \geq \delta_{\alpha,n}$ or $\delta_{b,n} \geq \delta_{\alpha,n}$*

The proof of this theorem may be found in Nepomuceno and Martins [6].

4. METHODOLOGY

Consider the following interval extensions of the logistic map (Eq. 2):

$$G(X) = rX(1 - X) \quad (3)$$

$$H(X) = r(X(1 - X)). \quad (4)$$

It may be observed that both are representations mathematically equivalent [15].

The computationally generated pseudo-orbits for 100 iterations of (3) and (4), $\{\hat{x}_{G,n}\}$ and $\{\hat{x}_{H,n}\}$, respectively, are considered for analysis, and, as already mentioned, for 0.4 as initial condition and $r = 3.8$. A third pseudo-orbit $\{\hat{x}_{P,n}\}$, from a interval extension $P(X)$ of (2), was obtained through the use of the variable-precision arithmetic (VPA) Matlab function.

All the computational tests were performed using the Matlab R2016a on an Intel i5-4440 CPU @ 3.10GHz and Windows 8 (64 bits) operating system.

5. RESULTS AND DISCUSSION

Figure (1) shows the results for the simulation of interval extensions from Equations (3) and (4) after 40 iterates. In the

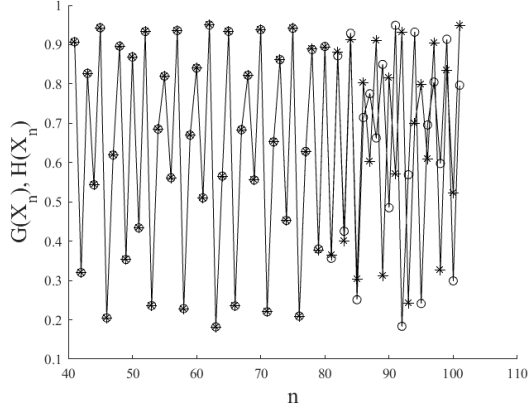


Figure 1 – Simulation of $G(X)$ and $H(X)$. Results of n iterates for $G(X_n)$ (—○—) and $H(X_n)$ (—*—), considering $X_0 = 0.4$ and $r = 3.8$.

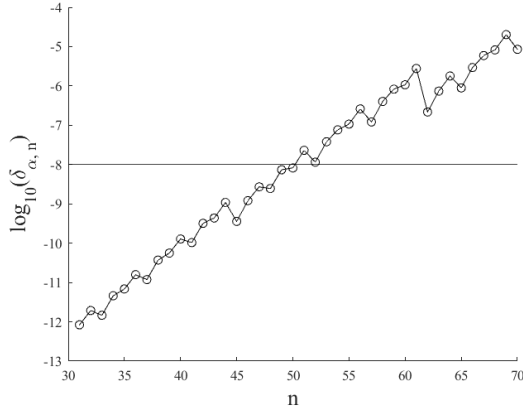


Figure 2 – Evolution of the lower bound error $\delta_{\alpha,n}$. Values plotted using the \log_{10} for $\delta_{\alpha,n}$.

51th iterate it may be already observed a lower bound error $\delta_{\alpha,51} = 10^{-7.638}$, greater than 10^{-8} (Figure 2). Thus, by Theorem 1, $\delta_{G,51} \geq 10^{-7.638}$ or $\delta_{H,51} \geq 10^{-7.638}$.

In Figure 3, results for the simulation of $G(X)$ e $H(X)$ are represented, and also, results produced by the iteration process of the logistic map using 1000-digit precision through VPA, $P(X_n)$.

Let $\delta_{GP,n}$ represents $|\hat{x}_{G,n} - \hat{x}_{P,n}|$ and $\delta_{HP,n}$ represents $|\hat{x}_{H,n} - \hat{x}_{P,n}|$. The evolution of $\delta_{GP,n}$ and $\delta_{HP,n}$ is shown in Figure 4.

As $\{\hat{x}_{P,n}\}$ is produced through symbolic computing of high precision, $\{\hat{x}_{P,n}\}$ is expected to be very close to a true orbit, at least for a number of iterates much lower than the number of iterates, which is the case. Then it is expected that $\delta_{P,n} \approx 0$, $\delta_{G,n} \approx \delta_{GP,n}$, and $\delta_{H,n} \approx \delta_{HP,n}$. Considering that $\delta_{P,43} \approx 0$, $\delta_{G,43} \approx 10^{-7.921}$ and $\delta_{H,43} \approx 10^{-7.954}$, for example.

Therefore, by this logic that combines the Theorem 1 and the use of symbolic computing, it may be said that the results

of both interval extensions $G(X)$ and $H(X)$ are far from being good representations of the true orbit of the logistic map for $X_0 = 0.4$ and $r = 3.8$ for $n > 100$. Thus, the theorem proved by [8] does not hold in this computer and software.

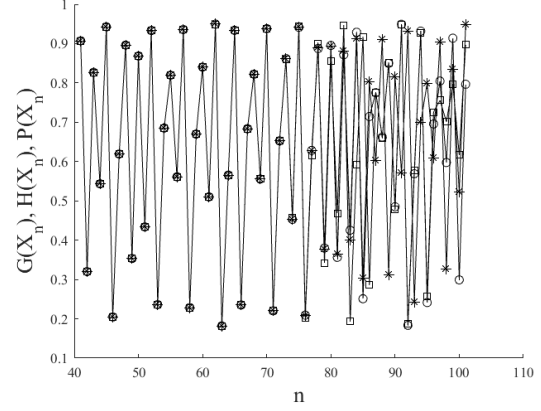


Figure 3 – Simulation of $G(X)$, $H(X)$ and $P(X)$. Results of n iterates for $G(X_n)$ (—○—), $H(X_n)$ (—*—) and $P(X_n)$ (—□—), considering $X_0 = 0.4$ and $r = 3.8$.

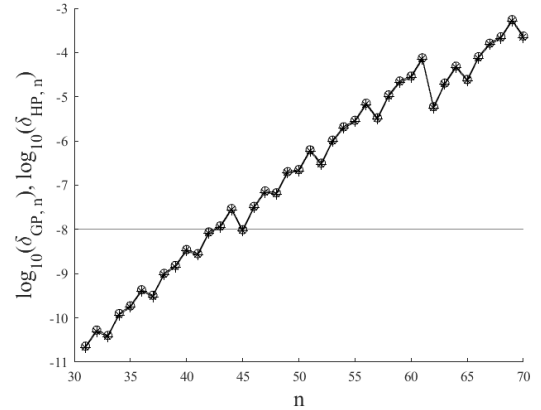


Figure 4 – Evolution of $\delta_{GP,n}$ and $\delta_{HP,n}$. Values plotted using the \log_{10} for $\delta_{GP,n}$ (—○—) and $\delta_{HP,n}$ (—*—).

6. CONCLUSION

From the presented results and discussion, it may be inferred that the shadowing property, as stated in [8] with $r = 3.8$ and $x_0 = 0.4$, cannot be associated to the logistic map simulations using the tested hardware and software that adopts the IEEE-754 floating-point standard.

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7. APPENDIX

7.1. Matlab code

```
clear all;
close all;
x=0.4;y=x;
r=3.8;
N=100;
%Pseudo-Orbits
for k=1:N
    x(k+1)=r*x(k)*(1-x(k));
    y(k+1)=r*(y(k)*(1-y(k)));
end
t=1:N+1;
figure(1)
id=41:101;
plot(t(id),x(id),'k-o',t(id),y(id),'-k*');
set(gca,'FontName','Times')
set(gca,'Box','off')
ylabel('G(X_n), H(X_n)', 'fontsize', 16);
xlabel('n', 'fontsize', 16);
%Symbolic - VPA
digits(1000);
z=vpa('4/10');
r=vpa('38/10');
for k=1:N
    z(k+1)=vpa(r.*z(k).(1-z(k)));
end
figure(2)
plot(t(id),x(id),'k-o',t(id),y(id),'-k*',...
     t(id),z(id),'-ks');
set(gca,'FontName','Times')
set(gca,'Box','off')
ylabel('G(X_n), H(X_n), P(X_n)', 'fontsize', 16);
xlabel('n', 'fontsize', 16);
for(k=1:100)
e(k) = abs(x(k)-y(k))/2;
end
for(k=1:100)
ex(k) = abs(x(k)-z(k));
end
for(k=1:100)
ey(k) = abs(y(k)-z(k));
end
figure(3)
ie=31:70;
plot(t(ie),log10(e(ie)),'ko-');
hold on;
plot([30,70],[-8,-8],'k-');
set(gca,'FontName','Times')
set(gca,'Box','off')
ylabel('log_{10}(\delta_{\alpha,n})',...
     'fontsize',16);
xlabel('n', 'fontsize', 16);
figure(4)
plot(t(ie),-8,'-',t(ie),log10(ex(ie)),'ko-',...
     t(ie),log10(ey(ie)),'k*-');
hold on;
plot([30,70],[-8,-8],'k-');
set(gca,'FontName','Times')
set(gca,'Box','off')
s1='log_{10}(\delta_{GP,n})';
s2='log_{10}(\delta_{HP,n})';
ylabel(strcat(s1,s2),'fontsize', 16);
xlabel('n', 'fontsize', 16);
```