

Computer Arithmetic

Master of Science in Electrical Engineering

Erivelton Geraldo Nepomuceno

Department of Electrical Engineering
Federal University of São João del-Rei

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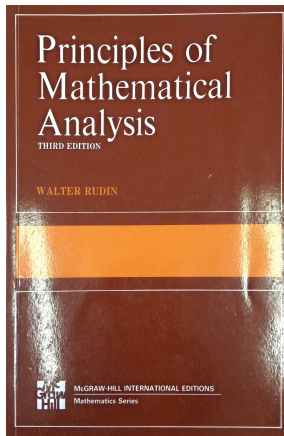
Teaching Plan

Content

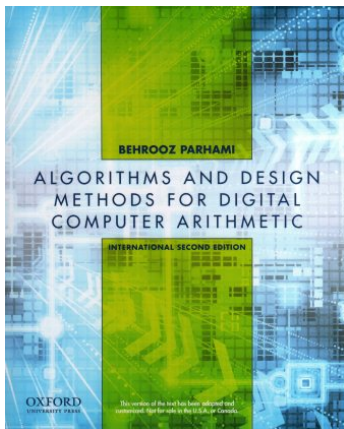
- 1 The Real and Complex Numbers Systems
- 2 Basic Topology
- 3 Numerical Sequences and Series
- 4 Continuity and Differentiation
- 5 Sequences and Series of Functions
- 6 Number Representation
- 7 Arithmetic Operation in a Computer
- 8 Programmable Logic Devices (FPGA)
- 9 IEEE 754-2008: Standard for Floating-Point Arithmetic
- 10 IEEE 1788-2008: Standard for Interval Arithmetic

References

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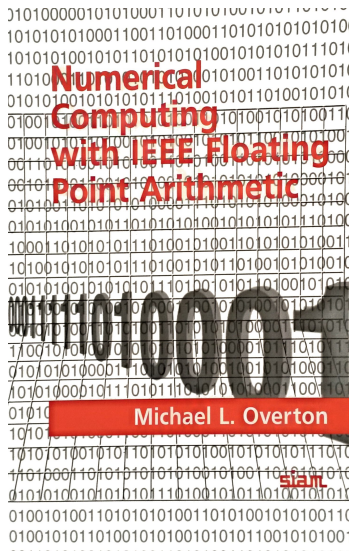
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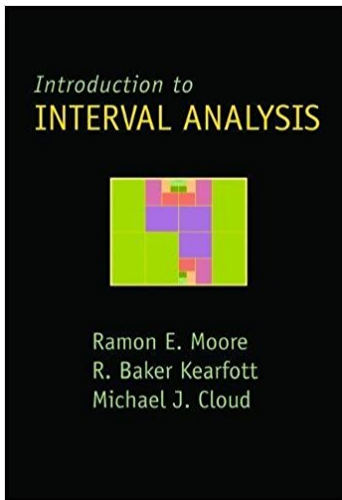
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Assessment

- $N_1 = 80$ points : Conference paper.
- $N_2 = 20$ points : Student engagement
- $N = N_1 + N_2$ points
- **If $N \geq 60$ then Succeed.**
- **If $N < 60$ then Failed.**

1. The real and complex number systems

1.1 Introduction

- A discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined **number concept**.
- **Number**: An arithmetical value expressed by a word, symbol, or figure, representing a particular quantity and used in counting and making calculations. (Oxford Dictionary).
- Let us see if we really know what a number is.
- Think about this question:¹

$$\text{Is } 0.999 \dots = 1? \quad (1)$$

¹Richman, F. (1999) Is $0.999 \dots = 1$? *Mathematics Magazine*. 72(5), 386–400.

- The set \mathbb{N} of **natural numbers** is defined by the Peano Axioms:
 - 1 There is an injective function $s : \mathbb{N} \rightarrow \mathbb{N}$. The image $s(n)$ of each natural number $n \in \mathbb{N}$ is called **successor** of n .
 - 2 There is a unique natural number $1 \in \mathbb{N}$ such that $1 \neq s(n)$ for all $n \in \mathbb{N}$.
 - 3 If a subset $X \subset \mathbb{N}$ is such that $1 \in X$ and $s(X) \subset X$ (that is, $n \in X \Rightarrow s(n) \in X$) then $X = \mathbb{N}$.
- The set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ of **integers** is a bijection $f : \mathbb{N} \rightarrow \mathbb{Z}$ such that $f(n) = (n - 1)/2$ when n is odd and $f(n) = n/2$ when n is even.
- The set $\mathbb{Q} = \{m/n; m, n \in \mathbb{Z}, n \neq 0\}$ of **rational numbers** may be written as $f : \mathbb{Z} \times \mathbb{Z}^* \rightarrow \mathbb{Q}$ such that $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ and $f(m, n) = m/n$.

- The rational numbers are inadequate for many purposes, both as a field and as an ordered set.
- For instance, there is no rational p such that $p^2 = 2$.
- An **irrational number** is written as infinite decimal expansion.
- The sequence 1, 1.4, 1.41, 1.414, 1.4142 ... tends to $\sqrt{2}$.
- What is it that this sequence *tends to*? What is an irrational number?
- This sort of question can be answered as soon as the so-called “real number system” is constructed.

Example 1

We now show that the equation

$$p^2 = 2 \tag{2}$$

is not satisfied by any rational p . If there were such a p , we could write $p = m/n$ where m and n are integers that are not both even. Let us assume this is done. Then (2) implies

$$m^2 = 2n^2. \tag{3}$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of (3) is divisible by 4, so that n^2 is even, which implies that n is even.

Thus the assumption that (2) holds thus leads to the conclusion that both m and n are even, contrary to our choice of m and n . Hence (2) is impossible for rational p .

- Let us examine more closely the Example 1.
- Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$.
- We shall show that A contains no largest number and B contains no smallest.
- In other words, for every $p \in A$ we can find a rational $q \in A$ such that $p < q$, and for every $p \in B$ we can find a rational $q \in B$ such that $q < p$.
- Let each rational $p > 0$ be associated to the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \quad (4)$$

and

$$q^2 = \frac{(2p + 2)^2}{(p + 2)^2}. \quad (5)$$

- Let us rewrite

$$q = p - \frac{p^2 - 2}{p + 2} \quad (6)$$

- Let us subtract 2 from both sides of (6)

$$\begin{aligned} q^2 - 2 &= \frac{(2p + 2)^2}{(p + 2)^2} - \frac{2(p + 2)^2}{(p + 2)^2} \\ q^2 - 2 &= \frac{(4p^2 + 8p + 4) - (2p^2 + 8p + 8)}{(p + 2)^2} \\ q^2 - 2 &= \frac{2(p^2 - 2)}{(p + 2)^2}. \end{aligned} \quad (7)$$

- If $p \in A$ then $p^2 - 2 < 0$, (6) shows that $q > p$, and (7) shows that $q^2 < 2$. Thus $q \in A$.
- If $p \in B$ then $p^2 - 2 > 0$, (6) shows that $0 < q < p$, and (7) shows that $q^2 > 2$. Thus $q \in B$.

- In this slide we show two ways to approach $\sqrt{2}$.
- Newton's method

$$\sqrt{2} = \lim_{n \rightarrow \infty} x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad (8)$$

which produces the sequence for $x_0 = 1$

Table 1: Sequence of x_n of (8)

n	x_n (fraction)	x_n (decimal)
0	1	1
1	$\frac{3}{2}$	1.5
2	$\frac{17}{12}$	1.41 $\bar{6}$
3	$\frac{577}{408}$	1.4142...

- Now let us consider the continued fraction given by

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots}}} \quad (9)$$

represented by $[1; 2, 2, 2, \dots]$, which produces the following sequence

Table 2: Sequence of x_n of (9)

n	x_n (fraction)	x_n (decimal)
0	1	1
1	3/2	1.5
2	7/5	1.4
3	17/12	1.41 $\bar{6}$...

Remark 1

The rational number system has certain gaps, in spite the fact that between any two rational there is another: if $r < s$ then $r < (r + s)/2 < s$. The real number system fill these gaps.

Definition 1

If A is any set, we write $x \in A$ to indicate that x is a member of A . If x is not a member of A , we write: $x \notin A$.

Definition 2

The set which contains no element will be called the **empty set**. If a set has at least one element, it is called **nonempty**.

Definition 3

If every element of A is an element of B , we say that A is a subset of B . and write $A \subset B$, or $B \supset A$. If, in addition, there is an element of B which is not in A , then A is said to be a **proper** subset of B .

1.2 Ordered Sets

Definition 4

Let S be a set. An **order** on S is a relation, denote by $<$, with the following two properties:

- 1 If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

- 2 If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

- The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold.

Definition 5

An **ordered set** is a set S in which an order is defined.

Definition 6

Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E . **Lower bound** are defined in the same way (with \geq in place of \leq).

Definition 7

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- 1 α is an upper bound of E .
- 2 If $\gamma < \alpha$ then γ is not an upper bound of E .

Then α is called the **least upper bound** of E or the **supremum** of E , and we write

$$\alpha = \sup E.$$

Definition 8

The **greatest lower bound**, or **infimum**, of a set E which is bounded below is defined in the same manner of Definition 7: The statement

$$\alpha = \inf E.$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E .

Example 2

If $\alpha = \sup E$ exists, then α may or may not be a member of E . For instance, let E_1 be the set of all $r \in \mathbb{Q}$ with $r < 0$. Let E_2 be the set of all $r \in \mathbb{Q}$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0,$$

and $0 \notin E_1$, $0 \in E_2$.

Definition 9

An ordered set S is said to have the **least-upper-bound property** if the following is true: If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

Theorem 1

Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then

$$\alpha = \sup L$$

exists in S and $\alpha = \inf B$.

1.3 Fields

Definition 10

A **field** is a set F with two operations, called **addition** and **multiplication**, which satisfy the following so-called “field axioms” (A), (M) and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F .
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F .
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.

- (M3) Multiplicative is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

Definition 11

An **ordered field** is a **field** F which is also an **ordered set**, such that

- 1 $x + y < x + z$ if $x, y, z \in F$ and $y < z$.
- 2 $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

1.4 The real field

Theorem 2

There exists an ordered field R which has the least-upper-bound property. Moreover, R contains Q as a subfield.

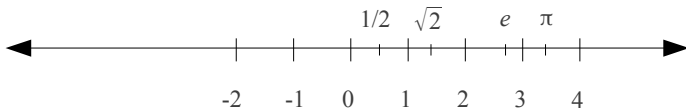


Figure 1: Real Line

Theorem 3

- (a) *If $x \in R$, and $x > 0$, then there is a positive integer n such that $nx > y$.*
- (b) *If $x \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

Theorem 4

For every real $x > 0$ and every integer $n > 0$ there is one and only one real y such that $y^n = x$.

Proof of Theorem 4:

- That there is at most one such y is clear, since $0 < y_1 < y_2$, implies $y_1^n < y_2^n$.
- Let E be the set consisting of all positive real numbers t such that $t^n < x$.
- If $t = x/(1 + x)$ then $0 < t < 1$. Hence $t^n < t < x$. Thus $t \in E$, and E is not empty. Thus $1 + x$ is an upper bound of E .
- If $t > 1 + x$ then $t^n > t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of E and there is $y = \sup E$.
- To prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to contradiction.

- The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}$$

when $0 < a < b$.

- Assume $y^n < x$. Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$

- Put $a = y$, $b = y + h$. Then

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

- Thus $(y + h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E .
- Assume $y^n > x$. Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \geq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

- Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E . But $y - k < y$, which contradicts the fact that y is the **least** upper bound of E .
- Hence $y^n = x$, and the proof is complete.

Definition 12

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$. Having chosen n_0, n_1, \dots, n_{k-1} , let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x.$$

Let E be the set of these numbers

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \quad (k = 0, 1, 2, \dots). \quad (10)$$

Then $x = \sup E$. The **decimal expansion** of x is

$$n_0 \cdot n_1 n_2 n_3 \dots \quad (11)$$

1.5 The extended real number system

Definition 13

The **extended real number system** consists of the real field R and two symbols: $+\infty$ and $-\infty$. We preserve the original order in R , and define

$$-\infty < x < +\infty$$

for every $x \in R$. An usual symbol for the extended real number system is \bar{R} .

- $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound.
- The same remarks apply to lower bounds.
- The extended real number system **does not form a field**.
- It is customary to make the following conventions:
 - (a) If x is real then

1.6 The complex field

(c) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

Definition 14

A **complex number** is an ordered pair (a, b) of real numbers. Let $x = (a, b)$, $y = (c, d)$ be two complex numbers. We define

$$x + y = (a + c, b + d),$$

$$xy = (ac - bd, ad + bc).$$

- $i = (0, 1)$.
- $i^2 = -1$.
- If a and b are real, then $(a, b) = a + bi$.

1.7 Euclidean Space

Definition 15

For each positive integer k , let R^k be the set of all ordered k -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_k),$$

where x_1, \dots, x_k are real numbers called the **coordinates** of \mathbf{x} .

- **Addition of vectors:** $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_k + y_k)$.
- **Multiplication of a vector by a real number (scalar):**
 $\alpha\mathbf{x} = (\alpha x_1, \dots, \alpha x_k)$.
- **Inner product:** $x \cdot y = \sum_{i=1}^k x_i y_i$.
- **Norm:** $|x| = (x \cdot x)^{1/2} = \left(\sum_{i=1}^k x_i^2\right)^{1/2}$.
- The structure now defined (the vector space R^k with the above product and norm) is called **Euclidean k -space**.

Theorem 5

Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$ and α is real. Then

- 1 $|\mathbf{x}| \geq 0$;
- 2 $|\mathbf{x}| = 0$ if and only if $|\mathbf{x} = \mathbf{0}|$;
- 3 $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;
- 4 $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;
- 5 $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- 6 $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{x} - \mathbf{z}|$.

- Items 1,2 and 6 of Theorem 5 will allow us to regard R^k as a **metric space**.

Exercises Chapter 1

- (1) Let the sequence of numbers $1/n$ where $n \in \mathbb{N}$. Does this sequence have an infimum? If it has, what is it? Explain your result and show if it is necessary any other condition.
- (2) Comment the assumption: Every irrational number is the limit of monotonic increasing sequence of rational numbers (Ferrar, 1938, p.20).
- (3) Prove Theorem 1.
- (4) Prove the following statements
 - a) If $x + y = x + z$ then $y = z$.
 - b) If $x + y = x$ then $y = 0$.
 - c) If $x + y = 0$ then $y = -x$.
 - d) $-(-x) = x$.

- (5) Prove the following statements
- a) If $x > 0$ then $-x < 0$, and vice versa.
 - b) If $x > 0$ and $y < z$ then $xy < xz$.
 - c) If $x < 0$ and $y < z$ then $xy > xz$.
 - d) If $x \neq 0$ then $x^2 > 0$.
 - e) If $0 < x < y$ then $0 < 1/y < 1/x$.
- (6) Prove the Theorem 2. (Optional)
- (7) Prove the Theorem 3.
- (8) Write addition, multiplication and distribution law in the same manner of Definition 22 for the complex field.
- (9) What is the difference between R and \bar{R} ?
- (10) Prove the reverse triangle inequality: $||a| - |b|| \leq |a - b|$.

2. Basic Topology

2.1 Finite, Countable, and Uncountable Sets

Definition 16

Consider two sets A and B , whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B , which we denote by $f(x)$. Then f is said to be a **function** from A to B (or a **mapping** of A into B). The set A is called the **domain** of f (we also say f is defined on A), and the elements of $f(x)$ are called the **values** of f . The set of all values of f is called the **range** of f .

Definition 17

Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the **image** of E under f . In this notation, $f(A)$ is the range of f . It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A **onto** B .

Definition 18

If $E \subset B$, f^{-1} denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the **inverse image** of E under f .

- f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

Definition 19

If there exists a 1-1 mapping of A onto B , we say that A and B , can be put in 1-1 **correspondence**, or that A and B have the same **cardinal number**, or A and B are equivalent, and we write $A \sim B$.

- Properties of equivalence
 - ▶ It is reflexive: $A \sim A$.
 - ▶ It is symmetric: If $A \sim B$, then $B \sim A$.
 - ▶ It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Definition 20

Let $n \in \mathbb{N}$ and J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A , we say:

- (a) A is **finite** if $A \sim J_n$ for some n .
- (b) A is **infinite** if A is not finite.
- (c) A is **countable** if $A \sim J$.
- (d) A is **uncountable** if A is neither finite nor countable.
- (e) A is **at most countable** if A is finite or countable.

Remark 2

A is infinite if A is equivalent to one of its proper subsets.

Definition 21

By a **sequence**, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes x_1, x_2, x_3, \dots . The values of f are called **terms** of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a **sequence in A** , or a **sequence of elements of A** .

- Every infinite subset of a countable set A is countable.
- Countable sets represent the “smallest infinity.”

Definition 22

Let A and Ω be sets, and suppose that with each element α of A is associated a subset of Ω which denote by E_α . A **collection of sets** is denoted by $\{E_\alpha\}$.

Definition 23

The **union** of the sets E_α is defined to be the set S such that $x \in S$ **if and only if** $x \in E_\alpha$ for **at least one** $\alpha \in A$. It is denoted by

$$S = \bigcup_{\alpha \in A} E_\alpha. \quad (12)$$

- If A consists of the integers $1, 2, \dots, n$, one usually writes

$$S = \bigcup_{m=1}^n E_m = E_1 \cup E_2 \cup \dots \cup E_n. \quad (13)$$

- If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \quad (14)$$

- The symbol ∞ indicates that the union of a **countable collection** of sets is taken. It should not be confused with symbols $+\infty$ and $-\infty$ introduced in Definition 13.

Definition 24

The **intersection** of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. It is denoted by

$$P = \bigcap_{\alpha \in A} E_\alpha. \quad (15)$$

- P is also written such as

$$P = \bigcap_{m=1}^n E_m = E_1 \cap E_2 \cap \cdots \cap E_n. \quad (16)$$

- If A is the set of all positive integers, we have

$$P = \bigcap_{m=1}^{\infty} E_m. \quad (17)$$

Theorem 6

Let $\{E_n\}$, $n = 1, 2, 3, \dots$, be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n. \quad (18)$$

Then S is countable.

- The set of all rational numbers is countable.
- The set of all real numbers is uncountable.

2.2 Metric Spaces

Definition 25

A set X , whose elements we shall call **points**, is said to be a **metric space** if with any two points p and q of X there is associated a real number $d(p, q)$ the **distance** from p to q , such that

- (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.
- (b) $d(p, q) = d(q, p)$;
- (c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Definition 26

By the **segment** (a, b) we mean the set of all real numbers x such that $a < x < b$.

Definition 27

By the **interval** $[a, b]$ we mean the set of all real number x such that $a \leq x \leq b$.

Definition 28

If $\mathbf{x} \in R^k$ and $r > 0$, the **open** (or **closed**) **ball** B with center at \mathbf{x} and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

Definition 29

We call a set $E \subset R^k$ **convex** if $(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \in E$ whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$ and $0 < \lambda < 1$.

Example 3

Balls are convex. For if $|\mathbf{y} - \mathbf{x}| < r$, $|\mathbf{z} - \mathbf{x}| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda\mathbf{y} + (1 - \lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda|\mathbf{y} - \mathbf{x}| + (1 - \lambda)|\mathbf{z} - \mathbf{x}| < \lambda r + (1 - \lambda)r \\ &= r. \end{aligned}$$

Definition 30

Let X be a metric space. All points and sets are elements and subsets of X .

- (a) A **neighbourhood** of a point p is a set $N_r(p)$ consisting of all points q such that $d(p, q) < r$.
- (b) A point p is a **limit point** of the set E if **every** neighbourhood of p contains a point $q \neq p$ such that $q \in E$.
- (c) If $p \in E$ and p is not a limit point of E , then p is called an **isolated point** of E .
- (d) E is **closed** if every limit point of E is a point of E .
- (e) A point p is an **interior point** of E if there is a neighbourhood N of p such that $N \subset E$.
- (f) E is **open** if every point of E is an interior point of E .
- (g) The **complement** of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

Definition 30

- (h) E is **perfect** if E is closed and if every point of E is a limit point of E .
- (i) E is **bounded** if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
- (j) E is **dense in X** if every point of X is a limit point of E , or a point of E (or both).

- If p is a limit point of a set E , then **every** neighbourhood of p contains **infinitely many** points of E .
- A set E is **open** if and only if **its complement is closed**.

Definition 31

If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X , then the **closure** of E is the set $\bar{E} = E \cup E'$.

Theorem 7

If X is a metric space and $E \subset X$, then

- (a) \bar{E} is closed.
- (b) $E = \bar{E}$ if and only if E is closed.
- (c) $E \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Theorem 8

Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

2.3 Compact Sets

Definition 32

By an **open cover** of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 33

A subset K of a metric space X is said to be **compact** if every open cover of K contains a **finite** subcover.

Definition 34

A set $X \subset \mathbb{R}$ is compact if X is closed and bounded^a.

^aLima, E. L. (2006) *Análise Real v. 1.*. RJ: IMPA, 2006.

Definition 35

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \dots$), then $\bigcap_1^\infty K_n$ is not empty.

Definition 36

If $\{I_n\}$ is a sequence of intervals in R^1 , such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3 \dots$), then $\bigcap_1^\infty I_n$ is not empty.

Theorem 9

If a set E in R^k has one of the following three properties, then it has the other two:

- 1 E is closed and bounded.
- 2 E is compact.
- 3 Every infinite subset of E has a limit point in E .

Theorem 10

(Weierstrass) Every bounded subset of R^k has a limit point in R^k .

2.4 Perfect Sets

Theorem 11

Let P be a nonempty perfect set in R^k . Then P is uncountable.

- Every interval $[a, b]$ ($a < b$) is uncountable. In particular, the set of all real numbers is uncountable.
- **The Cantor ternary set** is created by repeatedly deleting the open middle thirds of a set of line segments. One starts by deleting the open middle third $(1/3, 2/3)$ from the interval $[0, 1]$, leaving two line segments: $[0, 1/3] \cup [2/3, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. This process is continued ad infinitum, where the n th set is

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right). C_0 = [0, 1].$$

- The first six steps of this process are illustrated in Figure 50.



Figure 2: Cantor Set. Source: Wikipedia.

2.5 Connected Sets

Definition 37

Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A .

A set $E \subset X$ is said to be **connected** if E is **not** a union of two nonempty separated sets.

Theorem 12

A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Exercises Chapter 2

- (1) Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, be the set of real numbers y , such that $0 < y < x$. Complete the following statements
- (a) $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$.
 - (b) $\bigcup_{x \in A} E_x = E_1$.
 - (c) $\bigcap_{x \in A} E_x$ is empty.
- (2) Prove Theorem 6. Hint: put the elements of E_n in a matrix and count the diagonals.
- (3) Prove that the set of all real numbers is uncountable.
- (4) The most important examples of metric spaces are euclidean spaces R^k . Show that a Euclidean space is a metric space.

(5) For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned}d_1(x, y) &= (x - y)^2, \\d_2(x, y) &= \sqrt{|x - y|}, \\d_3(x, y) &= |x^2 - y^2|, \\d_4(x, y) &= |x - 2y|, \\d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}.\end{aligned}$$

Determine for each of these, whether it is a metric or not.

Work 1

To find the square root of a positive number a , we start with some approximation, $x_0 > 0$ and then recursively define:

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right). \quad (19)$$

Compute the square root using (19) for

- (a) $a = 2$;
- (b) $a = 2 \times 10^{-300}$
- (c) $a = 2 \times 10^{-310}$
- (d) $a = 2 \times 10^{-322}$
- (e) $a = 2 \times 10^{-324}$

Check your results by $x_n \times x_n$, after defining a suitable stop criteria for n .

3. Numerical Sequences and Series

3.1 Convergent Sequences

Definition 38

A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. In this case we also say that p_n converges to p , or that p is the limit of $\{p_n\}$, and we write $p_n \rightarrow p$, or

$$\lim_{n \rightarrow \infty} p_n = p.$$

- If $\{p_n\}$ does not converge, it is said to **diverge**.
- It might be well to point out that our definition of **convergent sequence** depends not only on $\{p_n\}$ but also on X .
- It is more precise to say **convergent in X** .
- The set of all points p_n ($n = 1, 2, 3, \dots$) is the **range** of $\{p_n\}$.
- The sequence $\{p_n\}$ is said to be **bounded** if its range is bounded.

Example 4

Let $s \in \mathbb{R}$. If $s_n = 1/n$, then

$$\lim_{n \rightarrow \infty} s_n = 0.$$

The range is infinite, and the sequence is bounded.

Example 5

Let $s \in \mathbb{R}$. If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.

Example 6

Let $s \in \mathbb{R}$. If $s_n = 1$ ($n = 1, 2, 3, \dots$), then the sequence $\{s_n\}$ converges to 1, is bounded, and has finite range.

Theorem 13

Let $\{p_n\}$ be a sequence in a metric space X .

- (a) $\{p_n\}$ converges to $p \in X$ if and only if every neighbourhood of p contains all but finitely many of the terms of $\{p_n\}$.
- (b) If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p' , then $p' = p$.
- (c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- (d) If $E \subset X$ and if p is a limit point of E , then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n \rightarrow \infty} p_n$.

Theorem 14

Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} t_n = t$. Then

(a) $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$;

(b) $\lim_{n \rightarrow \infty} cs_n = cs$, $\lim_{n \rightarrow \infty} (c + s_n) = c + s$, for any number c ;

(c) $\lim_{n \rightarrow \infty} (s_n t_n) = st$;

(d) $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$;

3.2 Subsequences

Definition 39

Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \dots$. Then the sequence $\{p_{n_i}\}$ is called a **subsequence** of $\{p_n\}$. If $\{p_{n_i}\}$, its limit is called a **subsequential limit** of $\{p_n\}$. It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p .

Theorem 15

- (a) *If $\{p_n\}$ is a sequence in a compact metric space X , then some subsequence of $\{p_n\}$ converges to a point of X .*
- (b) *Every bounded sequence in R^k contains a convergent subsequence.*

Theorem 16

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X .

3.3 Cauchy Sequence

Definition 40

A sequence $\{p_n\}$ in a metric space X is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.



Figure 3: Augustin-Louis Cauchy (1789-1857), French mathematician who was an early pioneer of analysis. Source: Wikipedia.

Definition 41

Let E be a subset of a metric space X , and let S be the set of all real number of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the **diameter** of E .

- If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \dots$, it is clear from the two preceding definitions that $\{p_n\}$ is a **Cauchy sequence if and only if**

$$\lim_{N \rightarrow \infty} \text{diam } E_N = 0.$$

Theorem 17

- (a) *If \bar{E} is the closure of a set E in a metric space X , then*

$$\text{diam } \bar{E} = \text{diam } E.$$

- (b) *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \dots$) and if*

Theorem 18

- (a) *In any metric space X , every convergent sequence is a Cauchy sequence.*
- (b) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X , then $\{p_n\}$ converges to some point X .*
- (c) *In R^k , every Cauchy sequence converges.*

- A sequence converges in R^k if and only if it is a Cauchy sequence is usually called the **Cauchy criterion** for convergence.

Definition 42

A sequence $\{s_n\}$ of real numbers is said to be

- (a) monotonically increasing if $s_n \leq s_{n+1}$ ($n = 1, 2, 3, \dots$);
- (b) monotonically decreasing if $s_n \geq s_{n+1}$ ($n = 1, 2, 3, \dots$);

3.4 Upper and Lower Limits

Theorem 19

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Definition 43

Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write $s_n \rightarrow +\infty$.

Definition 44

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers $x \in \bar{R}$ such that $s_{n_k} \rightarrow x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits plus possibly the numbers $+\infty$ and $-\infty$. Let $s^* = \sup E$, and $s_* = \inf E$. These numbers are called upper and lower limits of $\{s_n\}$.

- We can also write Definition 44 as

$$\limsup_{n \rightarrow \infty} s_n = s^*, \quad \liminf_{n \rightarrow \infty} s_n = s_*.$$

3.5 Some Special Sequences

- If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if $s_n \rightarrow 0$, then $x_n \rightarrow 0$. This property help us to compute the following the limit of the following sequences:

(a) If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and α is real, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$.

3.6 Series

Definition 45

Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^q a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where $s_n = \sum_{k=1}^n a_k$. For $\{s_n\}$ we also use the symbolic expression $a_1 + a_2 + a_3 + \cdots$ or, more concisely,

$$\sum_{n=1}^{\infty} a_n. \tag{20}$$

The symbol (33) we call an **infinite series**, or just a **series**.

- The numbers s_n are called the **partial sums** of the series.
- If $\{s_n\}$ converges to s , we say that the series converges, and we write

$$\sum_{n=1}^{\infty} a_n = s. \quad (21)$$

- s is the **limit of a sequence of sums**, and is not obtained simply by addition.
- If $\{s_n\}$ diverges, the series is said to diverge.
- Every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa.

- The Cauchy criterion can be restated as the following Theorem.

Theorem 20

$\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$\left| \sum_{k=n}^m a_k \right| \leq \varepsilon \quad (22)$$

if $m \geq n \geq N$.

Theorem 21

If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem 22

A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

- **Comparison test**

- (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.
- (b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

- **Geometric series**

- ▶ If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, the series diverges.

- ▶ **Proof** If $x \neq 1$, we have

$$s_n = \sum_{k=0}^n x^k = 1 + x + x^2 + x^3 \cdots + x^n. \quad (23)$$

If we multiply (23) by x we have

$$xs_n = x + x^2 + x^3 \cdots + x^{n+1}. \quad (24)$$

Applying (23)–(24) we have

$$\begin{aligned}S_n - xS_n &= 1 - x^{n+1} \\S_n(1 - x) &= 1 - x^{n+1} \\S_n &= \frac{1 - x^{n+1}}{1 - x}.\end{aligned}$$

The result follows if we let $n \rightarrow \infty$.

3.7 The Root and Ratio Tests

Theorem 23

(Root Test) Given $\sum a_n$, put $\alpha = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|}$. Then

- (a) If $\alpha < 1$, $\sum a_n$ converges;
- (b) If $\alpha > 1$, $\sum a_n$ diverges;
- (c) If $\alpha = 1$, the test gives no information.

Theorem 24

(Ratio Test) The series $\sum a_n$

- (a) converges if $\lim_{n \rightarrow \infty} \sup \left| \frac{a_{n+1}}{a_n} \right| < 1$,
- (b) diverges if $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for $n \geq n_0$, where n_0 is some fixed integer.

- The ratio test is frequently easier to apply than the root test. However, the root test has wider scope.

Exercises Chapter 3

- (1) Let $s \in R$. and $s_n = 1 + [(-1)^n/n]$. $\{s_n\}$ is bounded and its range is finite? Which value $\{s_n\}$ converges to?
- (2) Write a Definition for $-\infty$ equivalent to Definition 43.
- (3) Apply the root and ratio tests in the following series
 - (a) $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$,
 - (b) $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \dots$,

4. Continuity and Differentiation

4.1 Limits of Functions

Definition 46

Let X and Y be metric spaces: suppose $E \subset X$, f maps E into Y , and p is a limit point of E . We write $f(x) \rightarrow q$ as $x \rightarrow p$, or

$$\lim_{x \rightarrow p} f(x) = q \quad (25)$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon \quad (26)$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta. \quad (27)$$

- Alternative statement for Definition 46 based on (ε, δ) limit definition given by Bernard Bolzano in 1817. Its modern version is due to Karl Weierstrass ²

Definition 47

The function f approaches the limit L near c means: for every ε there is some $\delta > 0$ such that, for all x , if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

- f approaches L near c has the same meaning as the Equation (28)

$$\lim_{x \rightarrow c} f(x) = L. \quad (28)$$

²Adapted from Spivak, M. (1967) *Calculus*. Benjamin: New York.

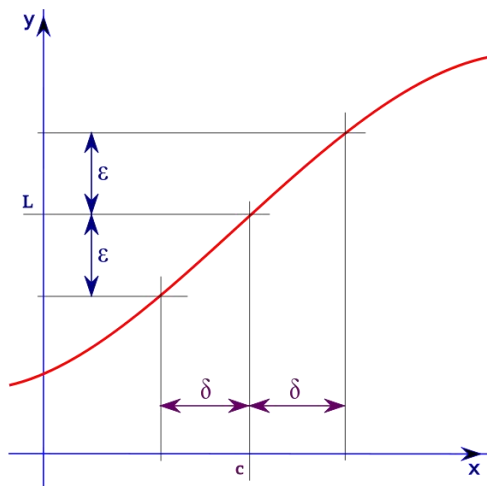


Figure 4: Whenever a point x is within δ of c , $f(x)$ is within ϵ units of L .
Source: Wikipedia.

Theorem 25

Let X, Y, E, f , and p be as in Definition 46. Then

$$\lim_{x \rightarrow p} f(x) = q \quad (29)$$

if and only if

$$\lim_{n \rightarrow \infty} f(p_n) = q \quad (30)$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \quad \lim_{n \rightarrow \infty} p_n = p. \quad (31)$$

Theorem 26

Suppose $E \subset X$, a metric space, p is a limit point of E , f and g are complex functions on E , and

$$\lim_{x \rightarrow p} f(x) = A, \quad \lim_{x \rightarrow p} g(x) = B.$$

Then

- (a) $\lim_{x \rightarrow p} (f + g)(x) = A + B;$
- (b) $\lim_{x \rightarrow p} (fg)(x) = AB;$
- (c) $\lim_{x \rightarrow p} \left(\frac{f}{g} \right) (x) = \frac{A}{B}, \quad \text{if } B \neq 0.$

4.2 Continuous Functions

Definition 48

Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y . Then f is said to be **continuous at** p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

- If f is continuous at every point of E , then f is said to be **continuous on** E .
- f has to be defined at the point p in order to be continuous at p .
- f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.

Theorem 27

Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y , g maps the range of f , $f(E)$, into Z , and h is the mapping of E into Z defined by

$$h(x) = g(f(x)) \quad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p . The function $h = f \circ g$ is called the composite of f and g .

4.3 Continuity and Compactness

Definition 49

A mapping \mathbf{f} of a set E into R^k is said to be **bounded** if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

Theorem 28

Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Theorem 29

Suppose f is a continuous real function on a compact metric space X , and

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p). \quad (32)$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

- The conclusion may also be stated as follows: There exist points p and q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$; that is, f attains its maximum (at p) and its minimum (at q).

Definition 50

Let f be a mapping of a metric space X into a metric space Y . We say that f is **uniformly continuous** on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon \quad (33)$$

for all p and q in X for which $d_X(p, q) < \delta$.

Theorem 30

Let f be a continuous mapping of a compact metric space X into a metric space Y . Then f is uniformly continuous on X .

4.4 Continuity and Connectedness

Theorem 31

If f is a continuous mapping of a metric space X into a metric space Y , and if E is a connected subset of X , then $f(E)$ is connected.

Theorem 32

(Intermediate Value Theorem) *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

4.5 Discontinuities

- If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is discontinuous at x .

Definition 51

Let f be defined on (a, b) . Consider any point x such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \rightarrow q$ as $n \rightarrow \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \rightarrow x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x) .

- It is clear that any point x of (a, b) , $\lim_{t \rightarrow x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \rightarrow x} f(t).$$

Definition 52

Let f be defined on (a, b) . If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the **first kind**. Otherwise, it is of the **second kind**.

4.6 Monotonic Functions

Definition 53

Let f be real on (a, b) . Then f is said to be **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$.

Theorem 33

Let f be monotonically increasing on (a, b) . Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b) . More precisely

$$\sup_{a < t < x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x < t < b} f(t). \quad (34)$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(x-). \quad (35)$$

4.7 Infinite Limits and Limits at Infinity

- For any real number x , we have already defined a neighborhood of x to be any segment $(x - \delta, x + \delta)$.

Definition 54

For any real c , the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Definition 55

Let f be a real function defined on E . We say that

$$f(t) \rightarrow A \text{ as } t \rightarrow x$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E, t \neq x$.

- Three important theorems.

Theorem 34

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$.

Theorem 35

If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that $f(x) \leq N$ for all x in $[a, b]$.

Theorem 36

If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$.

4.8 The Derivative of a Real Function

Definition 56

Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x), \quad (36)$$

and define

$$f'(x) = \lim_{t \rightarrow x} \phi(t), \quad (37)$$

provided this limit exists. f' is called the *derivative of f* .

Theorem 37

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x .

Theorem 38

Suppose f and g are defined on $[a, b]$ and are differentiable at point $x \in [a, b]$. Then $f + g$, fg and f/g are differentiable at x , and

$$(a) \quad (f + g)'(x) = f'(x) + g'(x);$$

$$(b) \quad (fg)'(x) = f'(x)g(x) + f(x)g'(x);$$

$$(c) \quad \left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)} \quad \text{with } g(x) \neq 0.$$

Theorem 5.1

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f , and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ and $(a \leq t \leq b)$, then h is differentiable at x , and

$$h'(x) = g'(f(x))f'(x). \quad (38)$$

Example 7

Let f be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases} \quad (39)$$

Applying the theorems, we have

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad (x \neq 0) \quad (40)$$

At $x = 0$ there is no $f'(x)$.

Definition 57

Let f be a real function defined on a metric space X . We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Theorem 39

Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

Theorem 40

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) , then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$.

Theorem 41

Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \gamma < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \gamma$.

Theorem 42

Suppose f and g are real and differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$, where $\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \rightarrow A \quad \text{as } x \rightarrow a. \quad (41)$$

If

$$f(x) \rightarrow 0 \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow a \quad (42)$$

or if

$$g(x) \rightarrow +\infty \text{ as } x \rightarrow a, \quad (43)$$

then

$$\frac{f(x)}{g(x)} \rightarrow A \text{ as } x \rightarrow a. \quad (44)$$

Definition 58

If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' the second derivative of f . Continuing in this manner, we obtain functions

$$f, f', f'', f^{(3)}, \dots, f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the n th derivative, or the derivative of order n , of f .

Theorem 43

Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k. \quad (45)$$

Example 8

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x \quad (46)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x \quad (47)$$