Complex Networks

Master of Science in Electrical Engineering

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2. Centrality Measures

- Centrality measures allow the key elements in a graph to be identified.
- The concept of centrality and the first related measures were introduced in the context of social network analysis.
- More recently have been applied to various other field.
- Measures of node centrality: degree centrality, the eigenvector centrality and the α-centrality.
- Centrality measures based on shortest paths: closeness centrality.

2.1 The Importance of Being Central

- Social networks analysis originated in the early 1920s, and focuses on relationships among social entities, such as communication and collaboration between members of a group, trades among nations, or economic transactions between corporations.
- This discipline is based on representing a social system as a graph whose nodes are the social individuals or entities, and whose edges represent social interactions.
- One of the primary uses of graph theory in social network analysis is the identification of the most important actors in a social network.

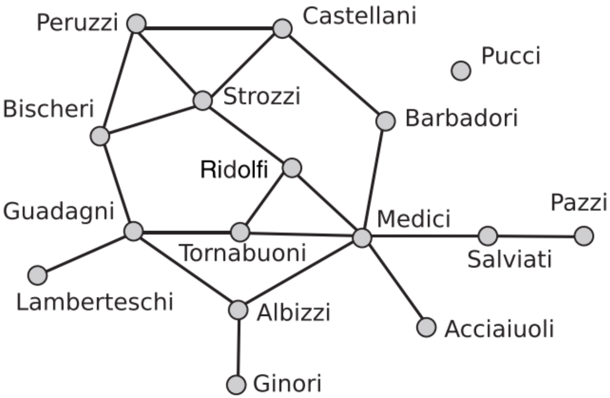


Figure 1: Marital relations between Florentine families.The graph has $N = 16$ nodes and $K = 20$. Each of the nodes in the graph is a family, and a link between a pair of families exists if a member of one family has married a member of the other.

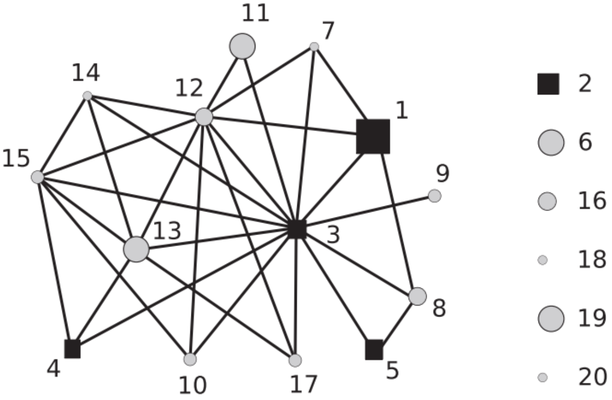


Figure 2: Graph of interactions in a group of primates. The graph, with $N = 20$ and $K = 31$, is an example of a social animal interaction network. It represents data recording three months of interactions among a group of 20 primates

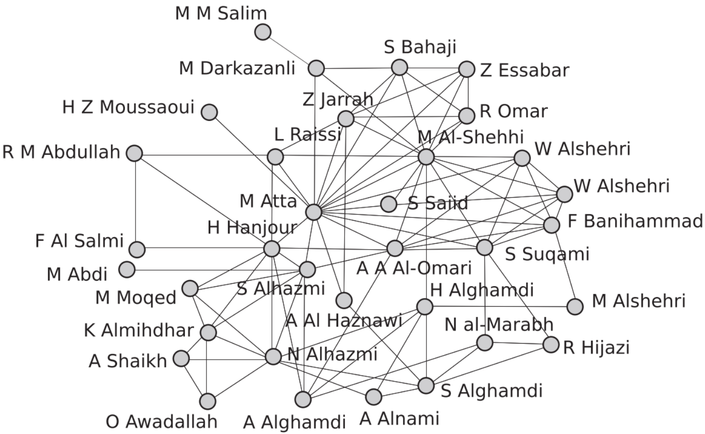


Figure 3: Contact network of hijackers and related terrorists of the September 2001 attacks. The graph with $N = 34$ nodes and $K = 93$. The nodes represent the 19 hijackers and 15 other associates who were reported to have had direct or indirect interactions with the hijackers.

Family	Wealth	Number of priorates	Node degree
Medici	103	53	6
Guadagni	8	21	4
Strozzi	146	74	4
Albizzi	36	65	3
Bischeri	44	12	3
Castellani	20	22	3
Peruzzi	49	42	3
Tornabuoni	48	..	3
Barbadori	55	..	2
Ridolfi	27	38	2
Salviati	10	35	2
Acciaiuoli	10	53	1
Ginori	32	..	1
Lamberteschi	42	0	1
Pazzi	48	..	1
Pucci	3	0	0

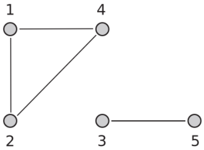
Figure 4: Two socio-economic indicators of the influence of a Florentine family, such as wealth and number of priorates, are correlated with the nodedegree of the family in the marriage network.

2.2 Connected Graphs and Irreducible Matrices

- A convenient mathematical way to describe the graphsis by means of the adjacency matrices.
- The structural properties of a graph turn into algebraic properties of its adjacency matrix.
- If the graph is connected, its adjacency matrix A is irreducible, while if the graph is unconnected, A is reducible.

Definition 1 (Reducible-irreducible matrix)

A matrix A is said to be reducible if there exists a $N \times N$ matrix P such that: $P'AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with A_{11} and A_{22} square matrices, and where P is a permutation matrix, i.e. a matrix such that each row and each column have exactly one entry equal to 1 and all others 0. Otherwise the matrix is irreducible.



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

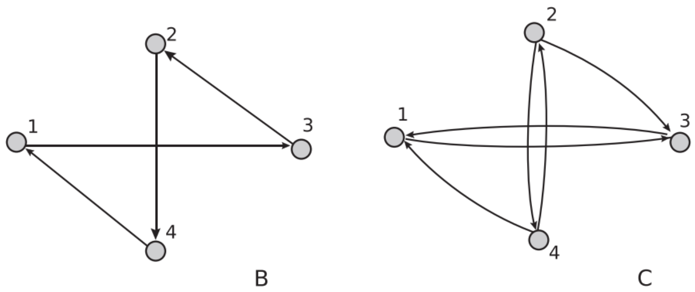
Figure 5: An undirected graph consisting of two components with, respectively, three and two nodes.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 6: $A' = P_{34}AP_{34}$. This is equivalent to writing $A' = P'_{34}AP_{34}$, since we have $P_{34} = P_{34}$. Matrix P_{34} is an elementary permutation matrix.

Theorem 1 (Adjacency matrix of connected graphs)

An undirected (directed) graph is connected (strongly connected) iff its adjacency matrix A is irreducible.



$$A_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 7: The two graphs, B and C, respectively corresponding to the adjacency matrices A_B and A_C.The first graph consists of a single strongly connected component, while the second one does not.

Theorem 2

A real non-negative $N \times N$ matrix A is irreducible iff: $(I + A)^{N-1} > 0$

- Since computer software is readily available to compute matrix powers, the expression $(I + A)^{N-1} > 0$ can be easily checked.
- If any entry of $(I + A)^{N-1}$ is zero, then the contrapositive of the theorem says A is reducible.

$$(I + A_B)^{N-1} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \end{pmatrix} \qquad (I + A_C)^{N-1} = \begin{pmatrix} 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \\ 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \end{pmatrix}$$

Figure 8: This proves that A_B is irreducible, while A_C is reducible, in agreement with the result found by visual inspection of the graphs B and C.

2.3 Degree and Eigenvector Centrality

- The number of neighbours of a node is a plausible measure of its structural centrality.
- This is because the higher the degree of a node, the more sources of information it has available, and the quicker the information will reach the node .
- We can therefore produce centrality rankings of the nodes of a graph based on the following definitions.

Definition 2 (Degree centralit)

In an undirected graph, the degree centrality of node i ($i = 1, 2, \ldots, N$) is defined as:

$$c_D^i = k_i = \sum_{j=1}^{N} a_{ij}, \tag{1}$$

where k_i is the degree of node i. The normalised degree centralities is given by

$$C_D^i = \frac{c_D^i}{N-1} \tag{2}$$

- A different way to generalise the degree centrality relies on the notion that a node can acquire high centrality either by having a high degree or by being connected to others that themselves are highly central.
- A simple possibility is to define the centrality c_i of a node i, with $i = 1, 2, \ldots, N$, as the sum of the centralities of its neighbours: $c_i = \sum_{j=1}^{N} a_{ij} c_j$. But it works only when the linear system is defined.

- In matrix notation Eq. (4) becomes

$$A\mathbf{c}^E = \lambda\mathbf{c}^E \tag{3}$$

- The standard way to overcome this problem is to define the centrality of a node as proportional to the sum of the centralities of its neighbours.
- Bonacich centrality or alternatively as the eigenvector centrality of node i:

$$\lambda c_i^E = \sum_{j=1}^{N} a_{ij} c_j^E \tag{4}$$

2.4 Other measures

- Measures based on shortest paths.
- Closeness centrality: based on the lengths of the shortest paths from a vertex
- Betweenness centrality: counting the number of shortest paths a vertex lies on
- Delta centrality: based both on the number and on the length of shortest paths.

Computational Exercise

1. Use the software Pajek and produce different visualisations of the Data Set 1 - Kindergarten.
2. Using only the visual information discuss with colleagues, which student is the most important according to your view of complex network.
3. Use the function `deg_seg` and check if the answer of (2) is correct.
4. Now use the function `pm` and discuss with your colleagues if your view of the most important student has changed.
5. At last, use `betweenness` to see how many times the shortest path has used the specific node. Divide the values by $(N-1)(N-2)$ to get the proportion.