Federal University of São João del-Rei
Department of Electrical Engineering
Electrical Engineering Graduate Program
CEFET-MG and UFSJ

Gabriel Oliveira Ferreira

State feedback control for linear and state polynomial continuous-time LPV systems under constraints

São João del-Rei
2021

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Master thesis presented to the Graduate Program in Electrical Engineering of the Federal University of São João del-Rei as part of requirements for obtaining the title of Master in Electrical Engineering.

Supervisor: Prof. Dr. Márcio Júnior Lacerda
Co-supervisor : Prof. Dr. Valter Júnior de Souza Leite

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Thank you all.

Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

Abstract

This work investigates new convex conditions to design robust and Linear Parameter Varying (LPV) gains for continuous time-varying systems subject to saturating actuator and energy bounded disturbances. The input-to-state stability conditions are used to design controllers ensuring the minimization of the $\mathcal{L}_2$-gain between the disturbance input and the controlled output. Furthermore, optimization procedures to maximize the estimate of the region of attraction, and the bound to the control signal as well, are formulated. By means of sum of squares (SOS) decomposition, this work is also concerned with the design of state-feedback controllers for LPV polynomial continuous-time systems. The vector field presents polynomial dependence on states. For both cases, the state-feedback gains designed depend on the filtered time-varying parameter. When designing time-varying controllers for time-varying systems in practical situations, the noise on the parameter measures may induce abrupt changes in the gain values and, consequently, in the control signal. Therefore, one may expect early damages on the actuator due to wear and fatigue. Hence, filtering the time-varying parameter and guaranteeing that the closed-loop filtered system is stable may produce state-feedback gains with smaller variance over time. The methods proposed consider the filtered parameter on the stabilization problems and, as an advantage over the existing literature, there is no need to know the bounds of the time-derivative of the parameter and such function does not need to be continuous. Convex conditions are also developed for the case when LPV polynomial continuous-time systems present input constraints. Hence, this work deals singly with three kinds of systems:

- input-to-state stabilization for continuous-time varying systems with saturating actuators;
- design of state-feedback controllers for continuous-time varying systems with state polynomial dependency.
- design of state-feedback controllers for continuous-time varying systems with state polynomial dependency and input constraints.

For all cases, the efficacy of the proposed methods is illustrated with numerical examples.

Key-words: Saturating actuators. Filtered parameter. Input-to-state. Region of attraction. Sum of squares. Polynomial continuous-time varying systems.

Contents

List of Figures

List of Tables

List of Acronyms and Notation

LMI	Linear Matrix Inequality .
LPV	Linear Parameter-Varying.
SOS	Sum of Squares.
M^T	transpose of matrix M.
$\mathtt{He}(M)$	denotes $M + M^T$ for any square matrix M.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}^n$	set of vectors of dimensions n with real elements.
ISS	Input-to-State.
$\mathbb{R}^{n \times m}$	set of matrices of dimensions $n \times m$ with real elements.
I_n	represents identity matrix of dimension n.
0_n	represents null matrix of dimension n.
$\mathcal{R}_{\mathcal{A}}$	represents the region of attraction.
$\mathcal{R}_{\mathcal{E}}$	represents the estimate of the region of attraction.
N	represents the number of vertices of the system.
$A \subseteq B$	A is a subset of B.
υ	denotes a small and strictly positive number.
$\mathcal{L}_2$	denotes the set of absolutely integrable signals.
ISS	Input-to-state stability.
$\Sigma\left[x\right]$	set of SOS decomposition.
$\|\|X(t)\|\|_2$	2-norm of $X(t)$.

Chapter 1

General introduction

In this work, conditions for the design of state-feedback controllers for linear and state polynomial continuous-time systems with time-varying parameters, subject to saturating actuators are developed. The approaches are based on LMIs (Linear Matrix Inequalities) for linear systems, and SOS (sum of squares) decomposition for the state polynomial dependent ones.

In this chapter, the effects of saturating actuators in linear systems are presented. A brief literature review about the control of continuous-time systems (linear and state polynomial ones) with saturating actuators is presented. At the end of the chapter, the main objectives of this work and the text organization are listed.

1.1 Problem Formulation

In this section, the problems that can emerge when dealing with saturating actuators are presented. The impacts of such nonlinearity are illustrated with examples, exhibiting how important is considering the maximum amplitude of control signal available when designing controllers for closed-loop systems.

1.1.1 Saturation: effects on closed-loop systems

Experimental systems present amplitude bound to their control signal, that is, the actuator has a physical limit, inside which is possible to guarantee that the closed-loop system trajectories converge to the origin or an input reference. This characteristic is called saturation and can bring many undesired effects, as limit cycle (in the phase plane, can be defined as an isolated closed curve (Slotine & Li, 1991)), performance degradation or instability. Besides, even for open-loop linear systems, the saturation makes the closed-loop ones nonlinear (Tarbouriech *et al.*, 2011a). To illustrate the impacts of the saturation, consider the following example.

Example 1.1 *Consider the continuous-time linear parameter varying (LPV) system borrowed from de Oliveira (2000):*

$$\begin{aligned}\dot{x}(t) &= A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t)\\ y(t) &= C(\theta(t))x(t) + D_u(\theta(t))u(t)\end{aligned} \tag{1.1}$$

where

$$A(\theta(t)) = \begin{bmatrix} -4.1 - 3\theta(t) & 1 \\ -2\theta(t) & 2 - 3.2\theta(t) \end{bmatrix},\ B_u(\theta(t)) = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$$B_w(\theta(t)) = \begin{bmatrix} -0.03 - 0.3\theta(t) \\ -0.47 + 0.9\theta(t) \end{bmatrix},\ C(\theta(t)) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ D(\theta(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$x(t) \in \mathbb{R}^n$ *is the state vector and* $u(t) \in \mathbb{R}^{n_u}$ *is the control signal vector.* $|\theta(t)| \leq 3$ *is the time-varying parameter. According to de Oliveira (2000),*

$$K = [-1.37885 \quad -9.96657]$$

is a state-feedback controller that stabilizes the system. Consider that ρ *is the maximum control signal amplitude available. Figure 1.1 shows the closed-loop system states trajectories when there is no saturation,* $|\rho| = \infty$*, and when* $|\rho| = 10$*. It is clear that, for both states, the setting time is bigger for the saturated system, exhibiting the impact of saturation on design requirements. Hence, such effect should be considered when designing controllers for experimental systems. The control signal is shown in Figure 1.2.*

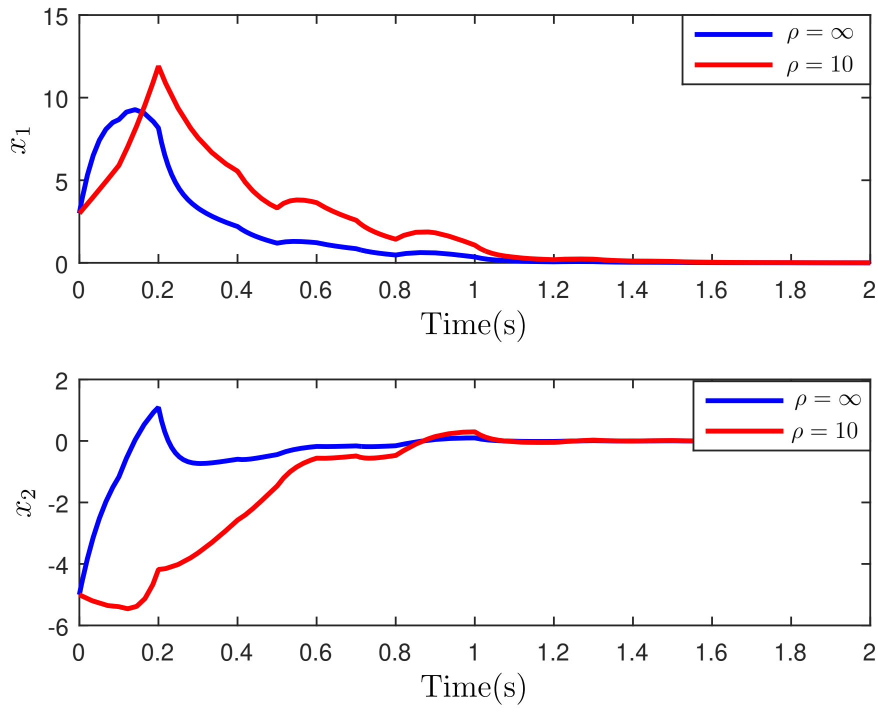

Figure 1.1: Saturation effects on design requirements.

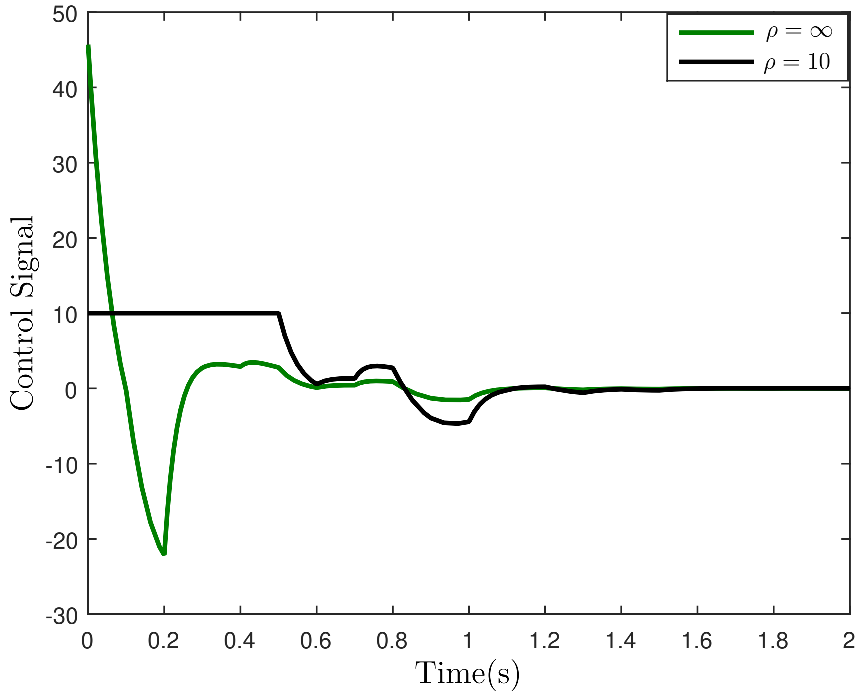

Figure 1.2: Control signal with amplitude saturation.

If the control signal amplitude bounds are set to smaller values, such as $\rho = 5$*, the saturation leads the closed-loop system to instability, as shown in Figure 1.3.*

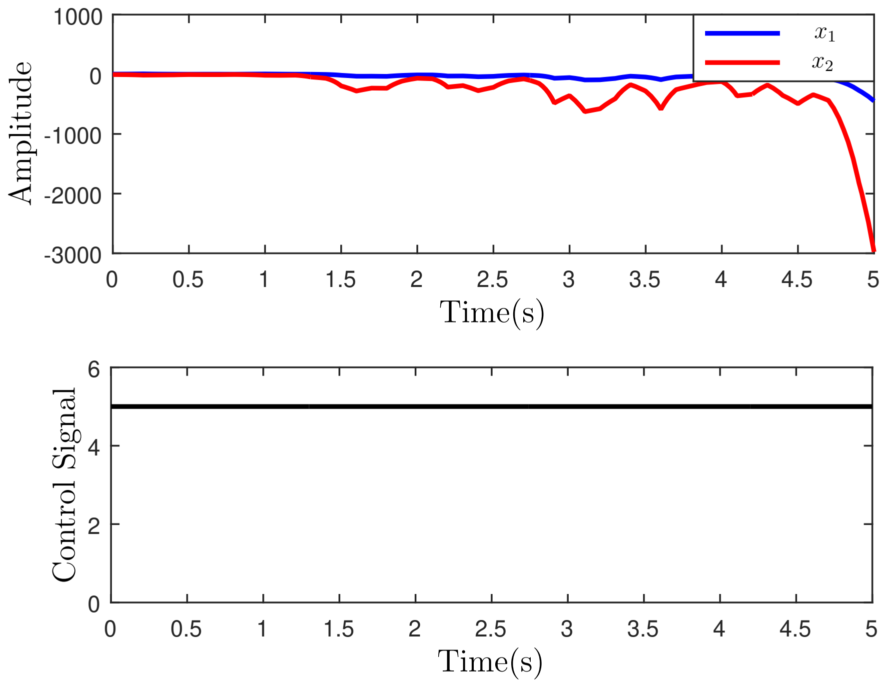

Figure 1.3: Saturation leading to an unstable closed-loop system.

The same effect can be noted for disturbance rejection. Figure 1.4 shows the closed-loop *system states trajectories when the disturbance*

$$w(t) = 20(\sin 2(t))e^{-0.1(t)}$$

is applied. As one may see, when $\rho = 9$, $x_1(t)$ and $x_2(t)$ present trajectories that go much farther from the origin when compared with $\rho = \infty$, that is, a disturbance rejection less effective.

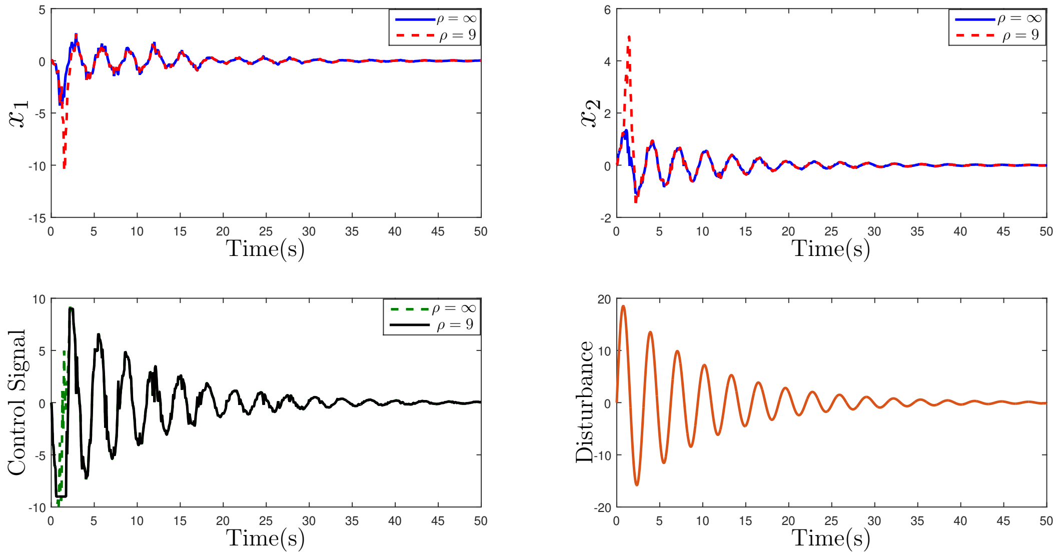

Figure 1.4: Disturbance rejection for different values of control signal available.

If $\rho = 5$, the closed-loop system becomes unstable for such disturbance, as shown in Figure 1.5.

1.2 Literature Review

LPV systems can be used to represent nonlinear ones, simplifying the search for stability certificates, and the design of filters and controllers. Currently, studies of LPV processes are on the rise, and several works can be found in the literature (Hanema *et al.*, 2020; Xiang, 2018; Yamamoto *et al.*, 2019; Zhang *et al.*, 2019; Briat, 2015a). A great part of the existing results has been developed by using the Lyapunov theory. These problems have been solved with the use of Linear Matrix Inequalities (LMIs). Such an approach has become very popular, much because of its relative simplicity and the increase in the number of specialized software dedicated to solving convex optimization problems.

One of the first methods employed on the study of LPV systems (Barmish, 1985; Horisberger & Belanger, 1976) consists in finding a Lyapunov function with a constant matrix that can guarantee the stability of the entire domain of the time-varying parameters. These results shed light on the solution to a great number of problems, although

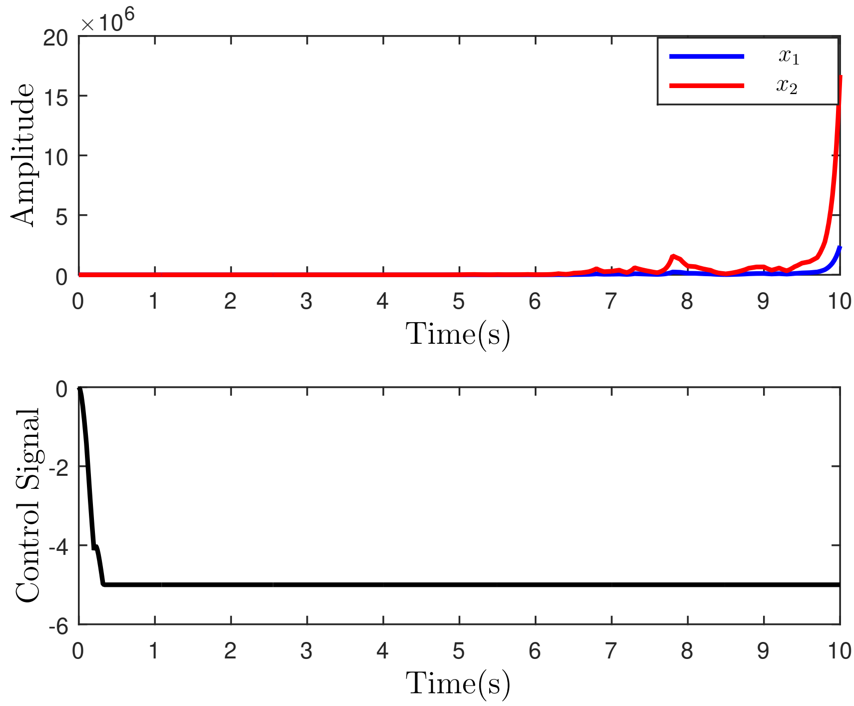

Figure 1.5: Saturation leading to instability for a disturbance response.

they may provide conservative results. To diminish the conservatism associated with the use of a single Lyapunov matrix, new methods (Chesi *et al.*, 2007; Oliveira & Peres, 2009; Peixoto *et al.*, 2020) based on the use of parameter-dependent Lyapunov functions have emerged.

The main challenge when considering parameter-dependent Lyapunov functions for LPV continuous-time systems is the computation of the time derivative of the Lyapunov matrix. Works in Geromel & Colaneri (2006); Montagner & Peres (2003) proposed LMI conditions to certify the stability of LPV continuous-time systems. The conditions consider the rates of variations of the time-varying parameters to have known bounds and belonging to a convex polytopic domain. This fact led to less conservative results when compared with quadratic stability conditions. In (Mozelli & Adriano, 2016), computational issues associated with solving parameter-dependent Lyapunov functions problems are discussed, exhibiting the growth of LMI number when the Lyapunov matrix depends on continuous time-varying parameters.

In Briat (2015b), LPV systems with piecewise constant parameters under constant and minimum dwell time are studied. The paper proposes conditions to certify stability for such systems, where the results are also extended to the problem of control design. In Briat & Mustafa (2017), the stability of LPV systems with differentiable parameters is investigated. The authors propose an approach based on the reformulation of LPV systems as their equivalent hybrid ones. LPV systems with piecewise constant parameters subject to spontaneous Poissonian jumps are studied in Briat (2018), where conditions for stability analysis and state-feedback control are developed. All these papers make use of the sum of squares decomposition to obtain finite-dimensional conditions.

A recent approach, employed originally in the context of Takagi-Sugeno fuzzy systems, consists of using a Lyapunov matrix depending on the filtered time-varying parameter (Cherifi *et al.*, 2019), where the authors show that it allows instantaneous jumps in the parameter, overcoming the inconvenience of bounding the time-derivative of the Lyapunov function.

Although LPV systems are extremely useful and relevant, it is well-known that such models are reliable just next to the linearization point. In this context, state polynomial systems can provide better results when modeling some physical processes. Many works are dealing with state-feedback design for state polynomial systems, providing closed-loop stability in a global as well as regional context (Valmórbida *et al.*, 2013) for time-invariant systems. Some results are based on state polynomial Lyapunov functions (Zhao & Wang, 2009) and others (Jennawasin & Banjerdpongchai, 2018) are based on state polynomial rational Lyapunov functions. In Zheng & Wu (2009), regional stabilization conditions were developed as state polynomial matrix inequalities and solved with the help of sum of squares (SOS) techniques. In that case, bounds to the derivatives of the state were used to solve the problem. In Ebenbauer & Allgöwer (2006), conditions for analysis and design of state polynomial control systems using dissipation inequalities are developed. The SOS method has also been employed in Wu & Prajna (2005); Gilbert *et al.* (2010), where models depending polynomially on the time-varying parameters were considered. It is also worthy of mention the work in Ebenbauer *et al.* (2005), that solves the stabilization problem by describing state polynomial systems as LPV ones.

Working with polynomial dependency, on states or time-varying parameters, may be a challenging task due to the size and complexity of the problem. Hence, considering this kind of system with constrained control conditions simultaneously, such as actuator saturation, makes finding solutions even harder. The design of controllers in the presence of saturating actuators makes necessary estimating the set of initial conditions, as well the maximum energy disturbance tolerable, such that the closed-loop system remains stable (de Souza *et al.*, 2019). Even for unstable linear open-loop systems, the actuator saturation makes the closed-loop nonlinear, and, for any state-feedback control law, the trajectories to the origin depend on the initial states (Tarbouriech *et al.*, 2011b).

There are many ways of treating the problem of actuator saturation and one known method is the modified sector condition, introduced by Gomes da Silva Jr. & Tarbouriech (2005) and used in Wang & Liu (2011) for discrete-time LPV systems. Concerning the saturating actuators, some interesting approaches presented in Groff *et al.* (2019a,b), are worthy of mentioning. In Groff *et al.* (2019a) the authors investigate precisely known and time-invariant systems with asymmetric saturating actuators using nonlinear sector condition and piecewise quadratic Lyapunov candidate functions. Also, piecewise affine systems are investigated in Groff *et al.* (2019b) using an implicit model given by the interconnection of a linear function and ramp nonlinearities, avoiding the explicit application of the nonlinear sector condition. In Delibaş *et al.* (2013), $\mathcal{L}_2$ control of continuous-time LPV systems with saturating actuators is studied. The authors use the modified sector condition to deal with the saturation problem and Pólya theorem to reduce the conservatism of the proposed method. The paper also discusses the relation between computational complexity and approach performance as the degree of the homogeneous polynomial and the Pólya's relaxation level are increased. Homogeneous polynomially parameter-dependent state-feedback controllers for LPV systems with saturating actuators are also studied in Montagner *et al.* (2007). The authors developed conditions to improve the estimates of the region of attraction (R.A.) for the closed-loop systems. The results obtained with parameter-dependent control laws are compared, in terms of the size of R.A., with robust controllers. The modified sector condition is used to deal with the saturating actuator. The control of continuous-time LPV systems subject to actuator saturation is also discussed in Nisha *et al.* (2018). The behavior of the saturation is represented by employing a convex hull approach.

1.3 Goals

The main objective of this work is designing state-feedback controllers for two classes of continuous-time systems with time-varying parameters and input saturation: linear and state polynomial ones. It is also developed controllers considering not only the system's initial conditions, but also input disturbances (Input to State Stability). Two different approaches are used: linear matrix inequalities, for linear systems, and sum of squares decomposition, for the state polynomial ones.

1.3.1 Specific Goals

- New synthesis conditions for LPV continuous-time systems subject to saturating actuators and energy bounded disturbances;
- new synthesis conditions for continuous-time varying systems with state polynomial dependency;
- design conditions for continuous-time varying systems with state polynomial dependency and input amplitude limitation.

- estimate the region of attraction for the cases where there are saturating actuators or input constraints.

1.4 Text organization

This text is divided in chapters for a better understanding of the steps developed during this work. Each chapter contains conditions to design state-feedback controllers for a specific kind of system, numerical experiments and final considerations.

Chapter 2

Presents the mathematical tools that are used throughout the text.

Chapter 3

Presents conditions to design state-feedback controllers for continuous-time systems with time-varying parameters (at least piecewise continuous), input saturation and subject to initial conditions and energy bounded disturbances. The $\mathcal{L}_2$-gain and some optimization problems, as minimum bound to the control signal and estimate of region of attraction, are also treated. The problem of tracking piecewise constant references is also discussed.

Chapter 4

Presents conditions to design state-feedback controllers for continuous-time systems with time-varying parameters (at least piecewise continuous) and polynomial dependency on the states. The approaches are based on the sum of squares decomposition (SOS) and the $\mathcal{L}_2$-gain is considered to give a robustness measure of the proposed conditions.

Chapter 5

Presents conditions to design state-feedback rational controllers for continuous-time varying systems with state polynomial dependency and input constraints. The approaches are also based on SOS decomposition and $\mathcal{L}_2$-gain is a design requirement considered.

Chapter 6

Presents the conclusion and the perspectives of future works.

Chapter 2

Mathematical tools

This chapter presents the mathematical tools that are used to develop the conditions proposed in the next chapters.

2.1 Continuous-time LPV systems

Consider the following LPV system

$$\dot{x}(t) = A(\alpha(t))x(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $\alpha(t)$ is the vector of time-varying parameters. It is possible writing such system as convex polytopic one, where $\alpha(t)$ belongs to the unit simplex. Matrix $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ depends linearly on $\alpha(t)$ and can be generically represented as

$$M(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t)M_i, \quad \alpha(t) \in \Lambda_N, \tag{2.2}$$

where M_i, $i = 1, \ldots, N$, are the vertices of the polytope and Λ_N is the unit simplex:

$$\Lambda_N = \big\{\alpha(t) \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i(t) = 1,\, \alpha_i(t) \geq 0,\, i = 1, \ldots, N\big\}. \tag{2.3}$$

2.2 Stability Analysis

The asymptotic stability of system (2.1) can be investigated by means of Lyapunov Theory. The following condition is known in the literature as Quadratic Stability (Barmish, 1985) and it is a sufficient stability condition for continuous time-varying systems.

Lemma 2.1 *If there exists a symmetric definite positive matrix P such that*

$$A_j^\top P + PA_j < \mathbf{0}, \quad j = 1, \ldots ,N, \tag{2.4}$$

then the system (2.1) *is asymptotically stable with a Lyapunov function described by*

$$V(x(t)) = x(t)^\top P x(t) > 0. \tag{2.5}$$

Proof: Choosing (2.5) as a Lyapunov Function and considering equation (2.1), one has

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^\top(t) P x(t) + x^\top(t) P \dot{x}(t) < 0, \\
\dot{V}(x(t)) &= x^\top(t) A(\alpha(t))^\top P x(t) + x^\top(t) P A(\alpha(t)) x(t) < 0, \\
\dot{V}(x(t)) &= x^\top(t) \big[A(\alpha(t))^\top P + P A(\alpha(t))\big] x(t) < 0.
\end{aligned}$$

As $A(\alpha(t))$ can be written as a convex polytope, the inequality above is equivalent to

$$\dot{V}(x(t)) = \sum_{j=1}^{N} x^\top(t) \big[A_j^\top P + P A_j\big] x(t) < 0,$$

$$V(x(t)) > 0 \quad \text{and} \quad \dot{V}(x(t)) < 0 \Longleftrightarrow P > \mathbf{0}, A_j^\top P + P A_j < \mathbf{0} \quad j = 1, \ldots, N.$$

■

2.3 $\mathcal{H}_\infty$ Norm

Consider a system whose state space representation is

$$\begin{aligned}
\dot{x}(t) &= A x(t) + B_w w(t) \\
z(t) &= C_z x(t) + D_w w(t), \quad x(0) = 0.
\end{aligned} \tag{2.6}$$

Its $\mathcal{H}_\infty$ norm is the maximum energy gain that the system can give to an input signal $w(t)$. So, it is possible writing the $\mathcal{H}_\infty$ norm according to

$$||H_{wz}(s)||_\infty = \max_{w \neq 0} \frac{||z(t)||_2}{||w(t)||_2} < \gamma, \quad w(t) \in \mathcal{L}_2, \tag{2.7}$$

where $\mathcal{L}_2$ denotes the set of absolutely integrable signals. Such value can be calculated by using LMIs. From (2.7), see that

$$||H_{wz}(s)||_\infty < \gamma \Longleftrightarrow z(t)^\top z(t) < \gamma^2 w(t)^\top w(t), \quad w(t) \in \mathcal{L}_2. \tag{2.8}$$

Hence, by choosing a Lyapunov function $V(x(t)) = x(t)^\top P x(t)$, considering a stable system, $x(0) = 0$ and imposing that

$$\dot{V}(x(t)) + z(t)^\top z(t) - \gamma^2 w(t)^\top w(t) < 0, \quad w(t) \in \mathcal{L}_2, \tag{2.9}$$

a symmetric positive definite P matrix that ensures the system stability and a value of γ can be obtained by considering equations (2.6) and (2.9), yielding

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA + C_z^\top C_z & P B_w + C_z^\top D_w \\ B_w^\top P + D_w^\top C_z & -\gamma^2 I_{n_w} + D_w^\top D_w \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < \mathbf{0}, \tag{2.10}$$

that take us to the well-known Bounded Real Lemma (Boyd *et al.*, 1994), guaranteeing the asymptotic stability of matrix A and an upper bound to γ:

$$\begin{bmatrix} A^\top P + PA + C_z^\top C_z & PB_w + C_z^\top D_w \\ B_w^\top P + D_w^\top C_z & -\gamma^2 I_{n_w} + D_w^\top D_w \end{bmatrix} < \mathbf{0}. \tag{2.11}$$

The value of the $\mathcal{H}_\infty$ norm can be obtained by solving the following convex optimization procedure:

$$\min_P(\gamma^2) : \begin{cases} P > 0, \\ \begin{bmatrix} A^\top P + PA + C_z^\top C_z & PB_w + C_z^\top D_w \\ B_w^\top P + D_w^\top C_z & -\gamma^2 I_{n_w} + D_w^\top D_w \end{bmatrix} < \mathbf{0}. \end{cases} \tag{2.12}$$

2.4 Filtered Time-Varying Parameter

Due to the assumptions on the time-varying parameter $\alpha(t)$, we may expect discontinuities and fast changes that could affect the performance of the actuators. Moreover, the discontinuities on $\alpha(t)$ impact the search for a Lyapunov candidate function, since unbound terms may arise in its time derivative. To overcome such issues, we propose to use a filtered version of $\alpha(t)$, leading to a smoother parameter signal. By hypothesis, we assume $\alpha(t)$ is (at least) piecewise continuous over the interval $[t-\beta, t]$, allowing to write for $\beta > 0$

$$\xi(t) = \beta^{-1} \int_{t-\beta}^{t} \alpha(\tau) d\tau, \quad \forall t > 0, \tag{2.13}$$

where $\xi(t)$ holds the convex sum property (Márquez *et al.*, 2016; Cherifi *et al.*, 2019):

$$\sum_{i=1}^{N} \xi_i = \sum_{i=1}^{N} \beta^{-1} \int_{t-\beta}^{t} \alpha(\tau) d\tau.$$

By using the linearity property, the integral and sum operators are interchangeable, leading to

$$\sum_{i=1}^{N} \xi_i = \beta^{-1} \int_{t-\beta}^{t} \sum_{i=1}^{N} \alpha_i(\tau) d\tau = \beta^{-1} \int_{t-\beta}^{t} d\tau = 1.$$

Moreover, from (2.13), the fundamental theorem of calculus allows us to write

$$\dot{\xi}(t) = \beta^{-1}\big(\alpha(t) - \alpha(t-\beta)\big). \tag{2.14}$$

Even for piecewise continuous functions, it is possible to see that $\dot{\xi}(t)$ is always finite and bounded for all $t \in [-\beta, +\infty]$. Another important aspect is that, for strictly positive values of β, $\xi(t)$ can be seen as a smoothed approximation of $\alpha(t)$. Furthermore, the bigger β, the smoother $\xi(t)$, as shown in Ferreira *et al.* (2020). The influence of such parameter is shown in Figure 2.1. The black signal is $\alpha(t)$ and its discontinuities are represented by the dashed lines. Such signal is filtered, generating the filtered parameter $\xi(t)$. Note that higher values of filtering taxes β lead to smoother signals, which is clear when one compares signals green and red and their respective values of β.

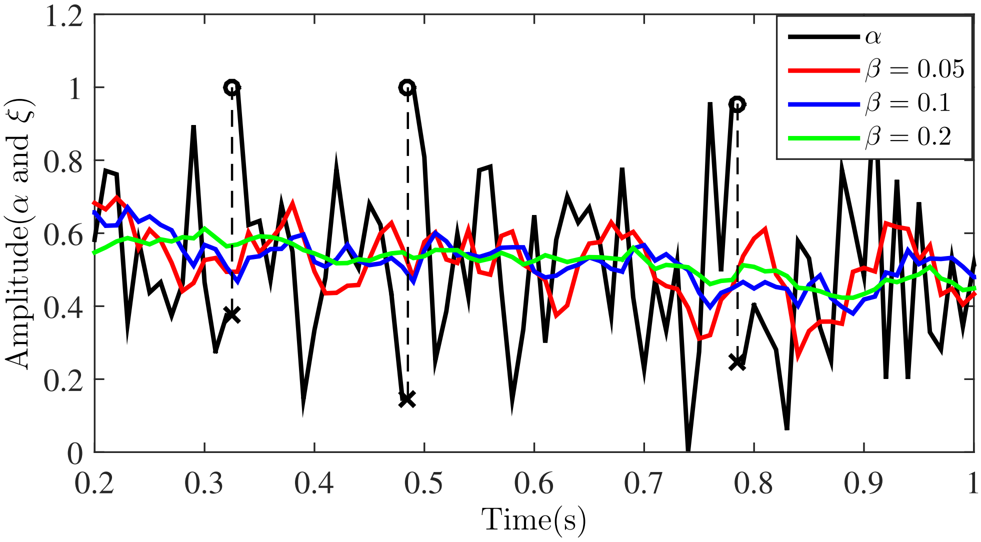

Figure 2.1: Measured parameter $\alpha(t)$ and its filtered version, $\xi(t)$, for different values of β.

2.5 Saturating Actuators

Consider the following continuous-time LPV system

$$x(t) = A(\alpha(t))x(t) + B(\alpha(t))\texttt{sat}\left(u(t)\right) \tag{2.15}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ represents the control signal vector without constraints and $\alpha(t) \in \Lambda_N$ is the time-varying parameter.

The saturating actuators impose on the system input magnitude bounds, given by $\rho \in \mathbb{R}^{n_u}$, that can be described as

$$\texttt{sat}(u(t))_{(\ell)} = \begin{cases} \rho_{(\ell)}, & \text{if} \quad u(t)_{(\ell)} > \rho_{(\ell)} \\ u(t)_{(\ell)}, & \text{if} \quad -\rho_{(\ell)} \leq u(t)_{(\ell)} \leq \rho_{(\ell)} \\ -\rho_{(\ell)}, & \text{if} \quad u(t)_{(\ell)} < -\rho_{(\ell)} \end{cases} \tag{2.16}$$

where ρ_ℓ is the input magnitude bound for each saturating actuator and $\ell = 1, \ldots n_u$. Note that this work treats the case when $\texttt{sat}(u(t))_{(\ell)}$ is symmetric, that is, the lower bound to the control signal has the same modulus than the upper one.

Consider that a state-feedback control law

$$u = K(\alpha(t))x(t), \tag{2.17}$$

where $K(\alpha(t)) \in \mathbb{R}^{n_u \times n}$ is applied on (2.15), than it can be rewritten according to

$$x(t) = A(\alpha(t))x(t) + B(\alpha(t))\texttt{sat}\left(K(\alpha(t))x(t)\right). \tag{2.18}$$

Equation (2.17) can also be applied on (2.16). Hence, the control signal available to the system can be represented according to Figure 2.2. As a consequence of the nonlinear

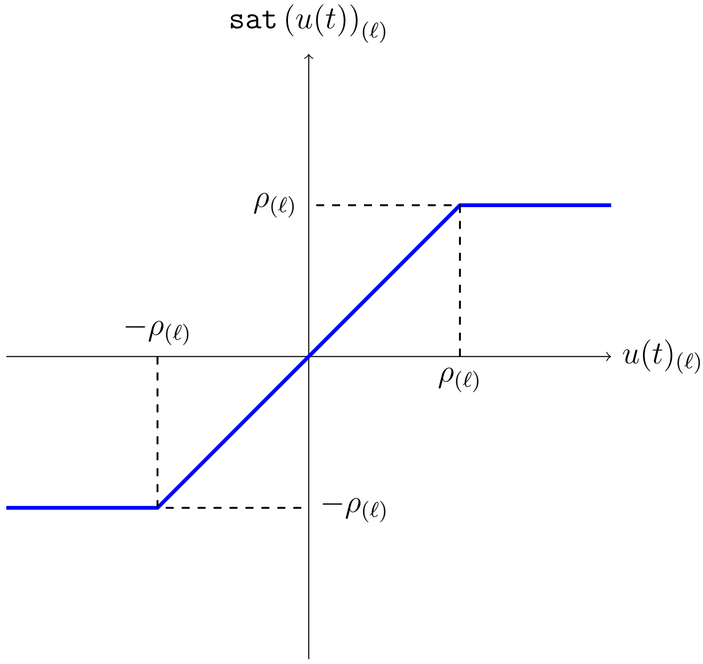

Figure 2.2: Saturation function.

behavior $sat(u(t))$, the closed-loop states trajectories to the origin depend on the system initial conditions. The set of all initial conditions such that the system trajectories converge to the origin is called basin or region of attraction, $\mathcal{R}_{\mathcal{A}}$. According to Tarbouriech *et al.* (2011c); Hu & Lin (2001), finding $\mathcal{R}_{\mathcal{A}}$ for many systems is not an easy task, because they can be open, unlimited or non-convex.

There are many ways of treating the saturation problem. Overview over Classical or Generalized Sector condition are presented in chapter 1 of Tarbouriech *et al.* (2011c). In this work, the Generalized Sector Condition is used to deal with the saturation. In such an approach, the saturating actuators are modeled in terms of the dead-zone nonlinearity, which can be defined as

$$\Psi(u(t)) = u(t) - \texttt{sat}\,(u(t))\,. \tag{2.19}$$

If the state-feedback control law (2.17) is applied on (2.15) and considering the dead-zone function in (2.19), the closed-loop system can be written as

$$\dot{x} = \mathcal{A}_{cl}(\alpha(t))x - B(\alpha(t))\Psi(u(t)) \tag{2.20}$$

where $\mathcal{A}_{cl} = A(\alpha(t)) + B(\alpha(t))K(\alpha(t))$ is the closed-loop dynamic matrix.

If we ensure a control signal $u(t)$ and an auxiliary signal $v(t) \in \mathbb{R}^{n_u}$ belonging to the set

$$\mathbb{S}(u(t) - v(t), \rho) = \left\{u(t) \in \mathbb{R}^n, v(t) \in \mathbb{R}^n : |u(t)_{(\ell)} - v(t)_{(\ell)}| \leq \rho_{(\ell)}\right\}, \tag{2.21}$$

$\forall \ell = 1, \ldots, n_u$, then, the following generalized sector condition concerning with the dead-zone nonlinearity (2.19)

$$\Psi(u(t))^\top S\Big(\Psi(u(t)) - v(t)\Big) \leq 0 \tag{2.22}$$

is verified for any positive definite diagonal matrix $S \in \mathbb{R}^{n_u \times n_u}$. We may choose $v(t)$ with similar structure to the signal $u(t)$, i.e., we consider $v(t) = G(\alpha(t))x(t)$, where G depends linearly on $\alpha(t)$, that is, an extra degree of freedom. The proof can be found in page 43 of Tarbouriech *et al.* (2011b) and Gomes da Silva Jr. & Tarbouriech (2005).

2.6 Input-to-State Stability

Consider a disturbance signal $w(t)$ such that

$$\mathcal{W} = \left\{ w(t) \in \mathbb{R}^{n_w} : \|w\|_2^2 \leq \delta^{-1} \right\}, \tag{2.23}$$

with

$$\|w\|_2^2 = \int_0^\infty w(t)^\top w(t) dt. \tag{2.24}$$

A closed-loop system is said input-to-state stable (ISS) (Sontag, 1999), if for all $w(t) \in \mathcal{W}$ and $x(0) \in \mathcal{R}_0$, the corresponding trajectories are bounded, i.e., they are confined in $\mathcal{R}_{\mathcal{A}}$ and if the disturbance is vanishing, then the trajectories converge asymptotically to the origin.

Consider the unstable time-invariant system described by,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u \mathtt{sat}\left(u(t)\right) + B_w w(t) \\ y(t) &= Cx(t) + D_w w(t) \end{aligned} \tag{2.25}$$

where

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \; B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ and } \quad D_w = [0]\,.$$

Consider that the control signal amplitude bound is $|\rho| \leq 1$ and $\delta^{-1} \leq 0.5$. Under these circumstances, a state-feedback controller (designed by solving theorem presented in next chapter) that makes the closed-loop system Input-to-State stable is given by $K = [1.9357 \quad -3.2506]$. Figure 2.3 shows that:

1. all trajectories initiated inside $\mathcal{R}_0$ remains inside $\mathcal{R}_{\mathcal{A}}$ for any $w(t) \in \mathcal{W}$ (green, black, gray and cyan lines). Note that even when $x(0)$ is on $\mathcal{R}_0$ border, the trajectories leave $\mathcal{R}_0$ but not $\mathcal{R}_{\mathcal{A}}$;

2. when $\delta^{-1} > 0.5$, even for $x(0) \in \mathcal{R}_0$, the closed-loop system can become unstable (dotted line: $\delta^{-1} = 56.25$);

3. even when $\delta^{-1} = 0$, system can become unstable if $x(0)$ is not inside $\mathcal{R}_{\mathcal{A}}$ (pink trajectory).

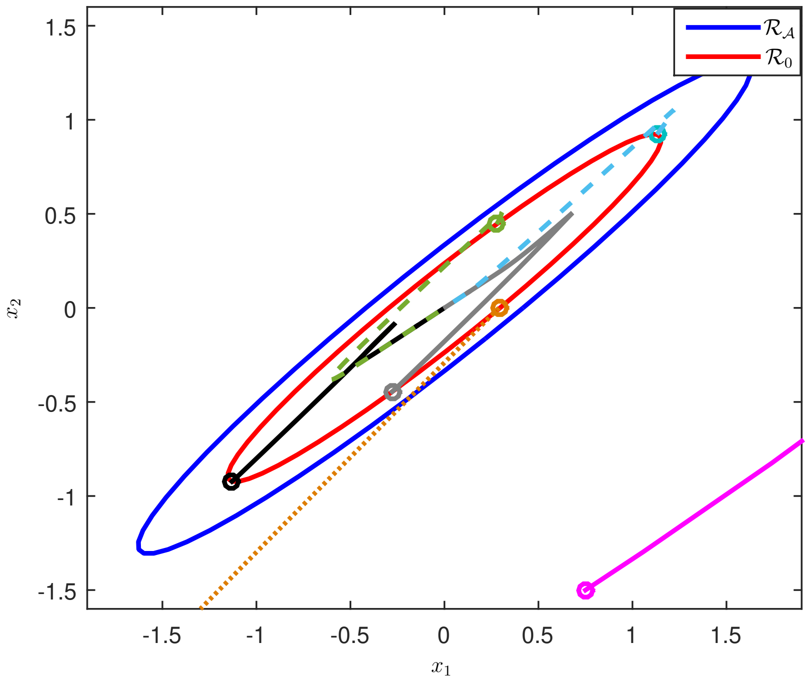

Figure 2.3: Input-to-State Stability.

In Figure 2.4, it is possible to see the amplitude of the disturbance applied to the closed-loop system and the behavior of both states through time. Besides, for a period of time, the control signal saturated. Hence, if one designs state-feedback controllers without considering saturation, there may be cases where the control signal is not enough to take the system's trajectory to the origin.

2.7 Sum of Squares

This work makes use of the sum of squares (SOS) decomposition to certify the non-negativity of the constraints arised from the Lyapunov theory. According to Wu & Prajna (2005), a multivariate polynomial $F(x_1,\ldots,x_n)$ of degree $2d$ is SOS if exist polynomials $f_1(x_1,x_2,\ldots,x_n),\ldots,f_m(x_1,x_2,\ldots,x_n)$, such that

$$F(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{m} f_i^2(x_1,x_2,\ldots,x_n), \tag{2.26}$$

and each f_i has degree lower or equal to d. It is clear that (2.26) is equal or greater than 0, allowing the SOS decomposition to be used as a non-negativity certificate. If there exists a SOS decomposition for $F(x_1,x_2,\ldots,x_n)$, then it can be written as

$$F(x_1,x_2,\ldots,x_n) = z^\top Q z, \tag{2.27}$$

where z is a vector of monomials with degree up to d in x and Q is a constant positive semi-definite matrix, which can be decomposed as $Q = V^\top V$. Then, all $f_i(x_1,x_2,\ldots,x_n)$

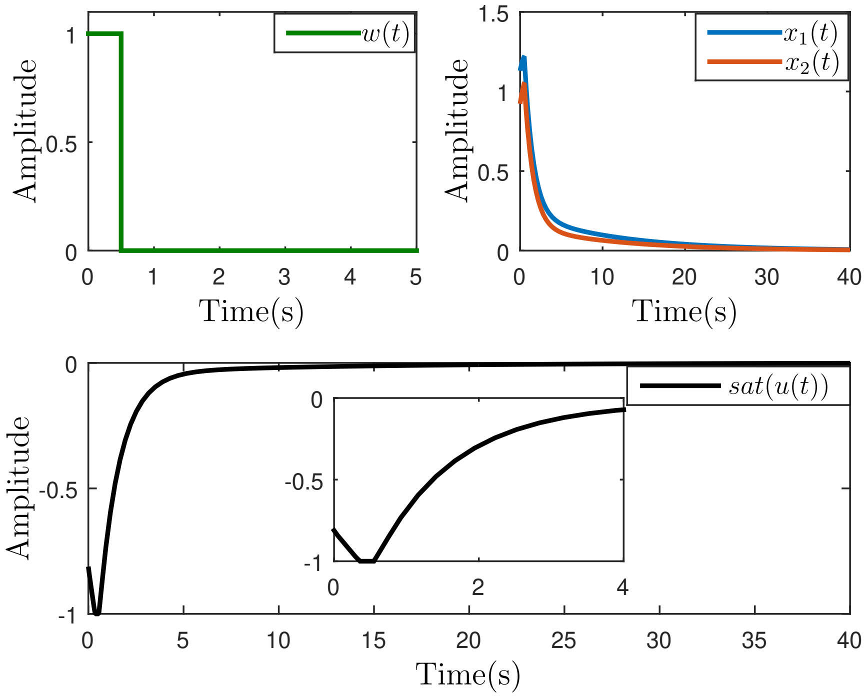

Figure 2.4: Input-to-State Stability: time response.

in (2.26) can be calculated according to

$$\begin{bmatrix} f_1(x_1,x_2,\ldots,x_n) \\ f_2(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_m(x_1,x_2,\ldots,x_n) \end{bmatrix} = V \begin{bmatrix} 1 & x_1 & \ldots & x_1^2 & x_1x_2 & \ldots & x_n^d \end{bmatrix}^\top . \tag{2.28}$$

To illustrate how a SOS decomposition can be found, consider the polynomial

$$F(x_1,x_2) = 4x_1^4 + 4x_1^3x_2 - 7x_1^2x_2^2 - 2x_1x_2^3 + 10x_2^4.$$

Note that $2d = 4$, so the vector z can be written as a function of monomials of degree $d = 2$, yielding

$$F(x_1,x_2) = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^\top \begin{bmatrix} 4 & 2 & -5 \\ 2 & 3 & -1 \\ -5 & -1 & 10 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}.$$

By employing a Cholesky factorization one has $Q = V^\top V$ with

$$V = \begin{bmatrix} 2 & 1 & -2.5 \\ 0 & \sqrt{2} & 0.75\sqrt{2} \\ 0 & 0 & \sqrt{2.625} \end{bmatrix}.$$

With the matrix V, all $f_i(x)$ of (2.26) can be calculated as

$$\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \\ f_3(x_1,x_2) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2.5 \\ 0 & \sqrt{2} & 0.75\sqrt{2} \\ 0 & 0 & \sqrt{2.625} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix},$$

which leads to

$$F(x_1,x_2) = f_1^2(x_1,x_2) + f_2^2(x_1,x_2) + f_3^2(x_1,x_2),$$

where $f_1(x_1,x_2) = 2x_1^2 + x_1x_2 - 2.5x_2^2$,
$f_2(x_1,x_2) = \sqrt{2}x_1x_2 + 0.75\sqrt{2}x_2^2$,
$f_3(x_1,x_2) = \sqrt{2.625}x_2^2$.

In this chapter, the mathematical tools necessary to develop the conditions proposed were presented.

Chapter 3

ISS under filtered parameters and saturating actuators

This chapter presents new convex conditions to design robust and LPV gains for continuous time-varying systems subject to saturating actuator and energy bounded disturbances. The input-to-state stability conditions are used to design controllers ensuring the minimization of the $\mathcal{L}_2$-gain between the disturbance input and the controlled output. Furthermore, optimization procedures to maximize the estimate of the region of attraction, and the bound to the control signal as well, are formulated. The efficacy of the proposed methods is illustrated with numerical examples, including a reference tracking problem, where comparisons with literature are also performed.

3.1 Problem Formulation

Problem 3.1 *Consider the continuous-time system under saturating actuators described by*

$$\begin{aligned} \dot{x}(t) &= A(\alpha(t))x(t) + B(\alpha(t))\texttt{sat}\,(u(t)) + B_w(\alpha(t))w(t) \\ y(t) &= C(\alpha(t))x(t) + D_w(\alpha(t))w(t), \end{aligned} \tag{3.1}$$

that depend linearly on the time-varying parameter $\alpha(t)$. Design a state-feedback controller depending linearly on smooth version of $\alpha(t)$, ensuring that:

1. *the closed-loop system, which will be presented in the following section, is said ISS.*
2. *The value of $\sqrt{\gamma}$, $\gamma > 0$, is an $\mathcal{L}_2$-gain between the exogenous input and the system output, thus verifying*

   $$\|y\|_2 = \sqrt{\gamma}(\|w\|_2 + \mathfrak{b}) \tag{3.2}$$

   for any $w \neq 0 \in \mathcal{W}$, and where the bias term $\mathfrak{b}$ is due to the non-null initial condition.

Associated with the items 1-2 of Problem 3.1, some optimization design procedures, that will be presented later, can be derived.

By considering a static feedback control law depending linearly on the parameter, as discussed in section 2.5, adapting equation (2.17) to

$$u = K(\xi(t))x(t), \tag{3.3}$$

with $K(\xi(t)) \in \mathbb{R}^{n_u \times n}$, and the dead-zone function $\Psi(u(t))$, it is possible to rewrite system's equation (3.1) as

$$\dot{x}(t) = (A(\alpha(t)) + B(\alpha(t))K(\xi(t)))x(t) - B(\alpha(t))\Psi(u(t)) + B_w(\alpha(t))w(t). \tag{3.4}$$

System (3.4) is nonlinear because of $\Psi(u(t))$. Hence, the local stability is required to characterize the allowed initial conditions ensuring their respective trajectories go to the origin. The set of all initial conditions resulting in trajectories that converge to the origin is denoted by $\mathcal{R}_{\mathcal{A}} \subseteq \mathbb{R}^n$, being called the region of attraction. Such a region can be non-convex, open, and even unbounded in some directions, making its characterization non trivial (Tarbouriech *et al.*, 2011b). Therefore, we provide an estimate region as ellipsoidal sets, $\mathcal{R}_{\mathcal{E}} \subseteq \mathcal{R}_{\mathcal{A}}$, as large as possible.

Moreover, because of Problem 3.1, we are also interested in a subset $\mathcal{R}_0 \subseteq \mathcal{R}_{\mathcal{E}}$, as large as possible, consisting of the initial conditions such that the respective closed-loop trajectories converge to the origin without leaving $\mathcal{R}_{\mathcal{E}}$, for a given disturbance $w(t)$ with bounded energy δ^{-1}, i.e., for $w(t) \in \mathcal{W}$.

We consider a non-quadratic Lyapunov function candidate, depending linearly on the parameter $\xi(t)$, given by

$$V(x(t), \xi(t)) = x(t)^\top P(\xi(t))^{-1} x(t). \tag{3.5}$$

Such a choice of structure for the Lyapunov candidate function allows us to keep a relatively low computational requirement while providing a solution for Problem 3.1 for a large class of systems. Other approaches in the literature use a version of (3.5) independent from the parameter and with a polynomial dependency on the state. In those cases, the computational burden may increase. Some works make use of to the SOS decomposition to deal with polynomial problems (on states or parameter), as in (Valmórbida *et al.*, 2013; Yang & Wu, 2014). Moreover, taking a parameter independent Lyapunov candidate function leads to a controller that can not be adapted to parameter changes, yielding to more conservative results.

To estimate the region of attraction, $\mathcal{R}_{\mathcal{E}}$, we use a level set of the Lyapunov function associated with the closed-loop system:

$$\mathcal{L}_{\mathcal{V}}(\mu) = \bigcap_{\forall\ \xi \in \Lambda} \mathcal{E}(P(\xi(t))^{-1}, \mu) \tag{3.6}$$

with $\mu > 0$ and the ellipsoidal set

$$\mathcal{E}(P(\xi(t)),\mu) = \{x(t) \in \mathbb{R}^n \mid x(t)^\top P(\xi(t))^{-1}x(t) \leq \mu^{-1}\}. \tag{3.7}$$

The level set presented in (3.6) remains as an infinite dimensional condition, since all values of $\xi(t)$ in the continuous set Λ must be verified. The following lemma, developed by Jungers & Castelan (2011) and where the reader can found the proof, presents an equivalent form to compute such a set through a finite dimensional condition.

Lemma 3.1 *Suppose that $V(x(t),\xi(t))$ is a Lyapunov function for system (3.4). Then, the level set (3.6) can be equivalently computed by using the finite dimensional condition*

$$\mathcal{L}_{\mathcal{V}}(\mu) = \bigcap_{\xi(t)\in\Lambda} \mathcal{E}(P(\xi(t))^{-1},\mu) = \bigcap_{i=1}^{N} \mathcal{E}(P_i^{-1},\mu) \tag{3.8}$$

for $\mu > 0$ and $\mathcal{E}(P_i^{-1},\mu) = \{x(t) \in \mathbb{R}^n; x(t)^\top P_i^{-1}x(t) \leq \mu^{-1}\}$.

3.2 Main Results

Theorem 3.1 *Consider a continuous-time system described by (3.1). If there exist symmetric positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $U_i \in \mathbb{R}^{n_u\times n}$, $Z_i \in \mathbb{R}^{n_u\times n}$, diagonal positive-definite matrices $S_i \in \mathbb{R}^{n_u\times n_u}$, with $i = 1,\dots,N$, and positive real scalars μ, and δ, such that*

$$\begin{bmatrix} \mathtt{He}\big(A_jP_i+B_jZ_i\big)-\frac{1}{\beta}(P_j-P_k) & -B_jS_i+U_i^\top & B_{wj} & P_iC_j^\top \\ \star & -S_i-S_i^\top & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_{wj}^\top \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{3.9}$$

$$\begin{bmatrix} P_i & Z_i^\top - U_i^\top \\ \star & \rho_{(\ell)}^2\mu \end{bmatrix} \geq \mathbf{0}, \tag{3.10}$$

$$\delta - \mu \geq 0 \tag{3.11}$$

hold, for all i, j, $k = 1,\dots,N$, and $\ell = 1,\dots,n_u$, then

$$K_i = Z_iP_i^{-1} \tag{3.12}$$

are such that the control law (3.3) with $K(\xi(t))$ computed as

$$K(\xi(t)) = \sum_{i=1}^{N} \xi_i(t)K_i \tag{3.13}$$

ensures a ISS closed-loop system, providing a solution to Problem 3.1, that is:

1. *for $w(t) = 0$, the trajectories of the closed-loop system initiated in the set $\mathcal{R}_{\mathcal{E}} = \mathcal{L}_{\mathcal{V}}(\mu)$, with $\mathcal{L}_{\mathcal{V}}$ given in (3.6) converge to the origin without leaving this set;*

2. *for any $w(t) \in \mathcal{W}$, $w(t) \neq 0$, the trajectories of system (3.4) remain confined in $\mathcal{R}_{\mathcal{E}} = \mathcal{L}_{\mathcal{V}}(\mu)$ for all initial conditions belonging to $\mathcal{R}_0 = \mathcal{L}_{\mathcal{V}}(\lambda)$, with $\lambda = (\mu^{-1} - \delta^{-1})^{-1}$;*

3. *for any $w(t) \in \mathcal{W}$, $w \neq 0$, and $x(0) \in \mathcal{L}_{\mathcal{V}}(\mu)$, $\sqrt{\gamma}$ is a guaranteed $\mathcal{L}_2$-gain between the exogenous input $w(t)$ and the output $y(t)$, ensuring (3.2) with $b = x(0)^\top P(\xi(t))^{-1}x(0)$.*

Note that $\mathtt{He}(M) = M^\top + M$.

Proof: The proof is given in two parts: one for the local ISS and $\mathcal{L}_2$-gain of the closed loop system, and another for the inclusion of the level set into $\mathbb{S}$. To simplify the proof notation, the dependence on t will be omitted for the time-varying parameters and states.

Stability part: Multiply (3.9) by $\xi_i(t)$, $i = 1,\ldots,N$, and sum it up to get the terms depending in i expressed in terms of $\xi(t)$. Additionally, replace $Z(\xi) = K(\xi)P(\xi)$, $U(\xi) = G(\xi)P(\xi)$, and $S(\xi) = T(\xi)^{-1}$, to get

$$\begin{bmatrix} \mathtt{He}\big(A_j + B_jK(\xi)P(\xi)\big) - \frac{1}{\beta}(P_j - P_k) & -B_jT(\xi)^{-1} + P(\xi)G(\xi)^\top & B_{wj} & P(\xi)C_j^\top \\ \star & -T(\xi)^{-1} - T(\xi)^{-T} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_{wj}^\top \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{3.14}$$

Next, multiply (3.14) by $\alpha_k(t-\beta)$, for $k = 1,\ldots,N$, $\alpha_j(t)$, for $j = 1,\ldots,N$, and sum it up to get,

$$\begin{bmatrix} \mathcal{R} & -B(\alpha)T(\xi)^{-1} + P(\xi)G(\xi)^\top & B_w(\alpha) & P(\xi)C(\alpha)^\top \\ \star & -T(\xi)^{-1} - T(\xi)^{-T} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_w(\alpha)^\top \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0},$$

with $\mathcal{R} = \mathcal{A}_{cl}P(\xi) + P(\xi)\mathcal{A}_{cl}^\top - \beta^{-1}\Big(P(\alpha) - P\big(\alpha(t-\beta)\big)\Big)$. By employing (2.14) and the convexity properties of the polytopic representation, one has

$$\begin{aligned} -\beta^{-1}\Big(P(\alpha) - P\big(\alpha(t-\beta)\big)\Big) &= -\sum_{i=1}^{N} \beta^{-1}\big(\alpha_i - \alpha_i(t-\beta)\big)P_i \\ &= \sum_{i=1}^{N} \dot{\xi}_i P_i = \dot{P}(\xi). \end{aligned}$$

Since $T(\xi)$ and $P(\xi)$ are regular matrices, we pre and post-multiply the last inequality by $\mathrm{diag}\,(P(\xi)^{-1}, T(\xi), \mathbf{I}, \mathbf{I})$. Moreover, we use the fact $P(\xi)P(\xi)^{-1} = I$ to write $\dot{P}(\xi)P(\xi)^{-1} + P(\xi)\dot{P}(\xi)^{-1} = \mathbf{0}$, or simply $-P(\xi)^{-1}\dot{P}(\xi)P(\xi)^{-1} = \dot{P}(\xi)^{-1}$, yielding

$$\begin{bmatrix} \mathrm{He}(P(\xi)^{-1}\mathcal{A}_{cl}) + \dot{P}(\xi)^{-1} & -P(\xi)^{-1}B(\alpha) + G(\xi)^\top T(\xi)^\top & P(\xi)^{-1}B_w(\alpha) & C^\top(\alpha) \\ \star & -T(\xi)^\top - T(\xi) & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & {D_w}^\top(\alpha) \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0},$$

By applying the Schur complement, and pre- and post-multiplying the resulting inequality by $\begin{bmatrix} x^\top & \Psi(u)^\top & w^\top \end{bmatrix}$ and its transpose, it is possible to get

$$\dot{V}(x) + \gamma^{-1} y^\top y - w^\top w - 2\Psi(u)^\top S(\xi)\Big(\Psi(u) - G(\xi)x\Big) < 0,$$

where $V(x) = x^\top P(\xi)^{-1} x$, and $\dot{V}(x) = \dot{x}^\top P(\xi)^{-1} x + x^\top P(\xi)^{-1}\dot{x} + x^\top \dot{P}(\xi)^{-1} x$. With $u = K(\xi)x$ and $v = G(\xi)x$, and supposing that these signals belong to the set $\mathbb{S}(u - v,\rho)$ given in (2.21), then the generalized sector condition (2.22) is ensured and it is possible to guarantee that

$$\dot{V}(x) + \gamma^{-1} y^\top y - w^\top w < 0,$$

meaning that with $\epsilon_1 = \max_{\xi\in\Lambda} \lambda\big(P(\xi)\big)$, $\epsilon_2 = \min_{\xi\in\Lambda} \lambda\big(P(\xi)\big)$ we verify

$$\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2.$$

Moreover, with $w = 0$ for $t \geq t_0$, we have

$$\dot{V}(x) \leq -\gamma^{-1} y^\top y + 2\Psi(u)^\top S(\xi)\Big(\Psi(u) - G(\xi)x\Big) \leq -\epsilon_3 \|x\|^2 < 0$$

for a small enough $\epsilon_3 > 0$. Thus, $V(x) = x^\top P(\xi)^{-1} x$ given in (3.5) is a Lyapunov function for the closed-loop system (3.4), allowing us to conclude on the local ISS stability of such a system.

Inclusion part: In the previous part, we required that signals u and v belong to the set $\mathbb{S}$. In this part of the proof, we demonstrate that the choice of initial conditions $x(0) \in \mathcal{L}_\mathcal{V}(1)$ ensures such a property.

If additionally to (3.9), the inequality (3.10) is also verified, multiply it by $\xi_i(t)$, $i = 1,\ldots,N$, add it up, and replace $Z(\xi)$ and $U(\xi)$ by $K(\xi)P(\xi)$ and $G(\xi)P(\xi)$, respectively, to write

$$\begin{bmatrix} P(\xi) & P(\xi)K(\xi)_{(\ell)}^\top - P(\xi)G(\xi)_{(\ell)}^\top \\ \star & \rho_{(\ell)}^2 \mu \end{bmatrix} \geq \mathbf{0}, \tag{3.15}$$

for all $\ell = 1, \ldots, n_u$. Pre- and post-multiplying the inequality (3.15) by $\operatorname{diag}\left(P(\xi)^{-1},1\right)$ and its transpose, and applying Schur complement, we get

$$P(\xi)^{-1} - \big[K(\xi)_{(\ell)} - G(\xi)_{(\ell)}\big]^\top \rho_{(\ell)}^{-2} \mu^{-1} \big[K(\xi)_{(\ell)} - G(\xi)_{(\ell)}\big] \geq \mathbf{0}$$

for all $\ell = 1, \ldots, n_u$. This last inequality can be pre- and post-multiplied by $x^\top$ and its transpose, respectively, yielding

$$x^\top P(\xi)^{-1} x \geq x^\top \big[K(\xi)_{(\ell)} - G(\xi)_{(\ell)}\big]^\top \rho_{(\ell)}^{-2} \mu^{-1} \big[K(\xi)_{(\ell)} - G(\xi)_{(\ell)}\big] x \tag{3.16}$$

for all $\ell = 1, \ldots, n_u$. With $u = K(\xi)x$, $v = G(\xi)x$, and taking the initial condition $x(0) \in \mathcal{L}_\mathcal{V}(\mu)$, we have that, due to the stability of the system,

$$x^\top P(\xi)^{-1}x \leq V(x(0)) = x(0)^\top P(\xi)^{-1}x(0) \leq \mu^{-1}.$$

Therefore, by choosing $x(0) \in \mathcal{R}_0$, (3.16) allows to write

$$1 \geq \mu V(x(0)) \geq \mu x^\top P(\xi)^{-1}x \geq |u_{(\ell)} - v_{(\ell)}|^2/\rho^2_{(\ell)},$$

which allows us to conclude that

$$|u_{(\ell)} - v_{(\ell)}| \leq \rho_{(\ell)}.$$

Thus, inequality (3.10) ensures that the generalized sector condition is verified with initial conditions belonging to $\mathcal{L}_\mathcal{V}(\mu)$. As a consequence, the level set contractiveness is also guaranteed, providing $\mathcal{R}_\mathcal{E} = \mathcal{L}_\mathcal{V}(\mu)$, meaning that the set $\mathbb{S}$ includes the contractive set $\mathcal{R}_\mathcal{E} = \mathcal{L}_\mathcal{V}$. As a consequence, the Lemma 3.1 is valid for any trajectory of the closed-loop system starting $\mathcal{L}_\mathcal{V}$ remains in $\mathbb{S}$. ∎

Consider the case when $w(t) = 0$ in (3.4). Then, Theorem 3.1 should be adapted according to the following corollary, to stabilize the system.

Corollary 3.1 *Consider a continuous-time system described by* (3.1)*. If there exist symmetric positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $U_i \in \mathbb{R}^{n_u\times n}$, $Z_i \in \mathbb{R}^{n_u\times n}$, diagonal positive-definite matrices $S_i \in \mathbb{R}^{n_u\times n_u}$, with $i = 1,\ldots,N$, and a positive scalar μ, such that*

$$\begin{bmatrix} \mathtt{He}\big(A_jP_i + B_jZ_i\big) - \frac{1}{\beta}(P_j - P_k) & -B_jS_i + U_i^\top \\ \star & -S_i - S_i^\top \end{bmatrix} < \mathbf{0}, \tag{3.17}$$

and (3.10) *hold, for all i, j, $k = 1,\ldots,N$, $r = 1,\ldots,n_u$, then the time-varying controller given by* (3.13) *ensures that the closed-loop system is asymptotically stable for all initial conditions belonging to $\mathcal{R}_\mathcal{E} = \mathcal{L}_\mathcal{V}(\mu)$.*

Note that if there is no information concerning the time-varying parameter is available, one may use $S_i = S$, $Z_i = Z$, and $P_i = P$, leading to a robust controller design.

Consider the case where no information about the time-varying parameter and its derivative is available. Then a special case of Theorem 3.1, presented on the following corollary, should be used to stabilize the system.

Corollary 3.2 *Consider a continuous-time system described by* (3.1)*. If there exist a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$, matrices $U \in \mathbb{R}^{n_u\times n}$ and $Z \in \mathbb{R}^{n_u\times n}$, a diagonal positive-definite matrix $S \in \mathbb{R}^{n_u\times n_u}$, and positive scalars μ and δ such that*

$$\begin{bmatrix} \mathtt{He}\big(A_jP + B_jZ\big) & -B_jS + U^\top & B_{wj} & PC_j^\top \\ \star & -S - S^\top & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{I} & D_{wj}^\top \\ \star & \star & \star & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{3.18}$$

$$\begin{bmatrix} P & Z^\top - U^\top \\ \star & \rho_{(\ell)}^2 \mu \end{bmatrix} > \mathbf{0}, \tag{3.19}$$

$$\delta - \mu > 0, \tag{3.20}$$

hold for all $j = 1, \ldots, N$ and $\ell = 1,...,n_u$, then the control law $u = Kx$ with $K = ZP^{-1}$ ensures a ISS closed-loop system, providing a solution to Problem 3.1, that is :

1. *for $w(t) = 0$, the trajectories initiated in the set $\mathcal{R}_\mathcal{E}$ converge to the origin within its domain, considering $\mathcal{R}_\mathcal{E} = \mathcal{L}_\mathcal{V}(\mu)$.*
2. *for any $w(t) \in \mathcal{W}$, $w(t) \neq 0$ the trajectories of system* (3.4) *remain inside $\mathcal{L}_\mathcal{V}(\mu)$ for all initial conditions that belong to $\mathcal{L}_\mathcal{V}(\lambda)$, where $\lambda = (\mu^{-1} - \delta^{-1})^{-1}$.*
3. *for any $w(t) \in \mathcal{W}$, $w(t) \neq 0$, and $x(0) \in \mathcal{L}_\mathcal{V}(\mu)$, $\sqrt{\gamma}$ is a guaranteed $\mathcal{L}_2$-gain between the exogenous input w and the output y, ensuring* (3.2) *with $\mathfrak{b} = x(0)^\top P^{-1} x(0)$.*

The proof of both corollaries follow the same steps as the proof of Theorem 3.1, making only the necessary changes.

3.2.1 Optimization Design Procedures

The proposed conditions work with different scalar variables, μ, δ, γ, and ρ, that can be optimized to specific performances of the closed-loop system. For instance, it is possible to design controllers that maximize $\mathcal{R}_\mathcal{E}$ or the maximal tolerable energy disturbance, δ^{-1}. Another design objective may be the minimization of the $\mathcal{L}_2$ gain, $\sqrt{\gamma}$, or the minimal saturating bound of the control signal that ensures the regional stability or a desired $\mathcal{L}_2$. Such objectives can be pursued with the aid of the convex optimization procedures shown discussed in the sequence.

Maximization of $\mathcal{R}_\mathcal{E}$: The design of the parameter-dependent control gain can be optimized to achieve a larger set of allowed initial conditions. Thus, the goal is to maximize the estimated region of attraction considering $w = 0$.

For maximizing the region of attraction, we consider the optimization problem given by maximizing an ellipsoid given by $\mathcal{E}(H,\mu)$, $H \in \mathbb{R}^{n\times n}$, such that $\mathcal{E}(H,\mu) \subset \mathcal{L}_\mathcal{V}(\mu)$, which can ensured by

$$\mathcal{T}_1 : \begin{cases} \min\limits_{P_i,U_i,Z_i} & \texttt{trace}(H) \\ \text{subject to:} & \text{Theorem 3.1 or Corollaries 3.2 or 3.1, and} \\ & \begin{bmatrix} P_i & \mathbf{I} \\ \star & H \end{bmatrix} > \mathbf{0}. \end{cases} \tag{3.21}$$

Other optimization procedures can be proposed in the form of

$$\mathcal{T}_2(f) : \begin{cases} \min f \\ \text{subject to:} \quad \text{Theorem 3.1 or Corollaries 3.2, 3.1} \end{cases} \tag{3.22}$$

where the objective function f can be adequately chosen with the intents of the designer. For instance, consider the following three cases:

1. Choose $f = \rho$ to search for the smallest amount of control signal necessary to stabilize the system and reject the disturbance.

2. Choose $f = \gamma$ to minimize the $\mathcal{L}_2$-gain between the disturbance w and the output y. Note that, if null initial conditions are employed, i.e., $x(0) = 0$, then $\delta^{-1} = \mu^{-1}$.

3. Choose $f = \mu$ to maximize the energy bound of the disturbance, δ^{-1}, for initial conditions $x(0) \in \mathcal{R}_0$. Note that if $x(0) = 0$, $\delta^{-1} = \mu^{-1}$.

Corollary 3.1 cannot be used in cases 2 and 3 because it is assumed there is no external disturbance, in such a case.

3.3 Numerical Experiments

This section illustrates the results for the design of stabilizing controllers for LPV continuous-time systems with saturating actuators. The effects of the filtered time-varying parameter ξ in the control design and some optimization design procedures will be explored. The routines were implemented in MATLAB, version 8.5.0.197613 (R2015a), Windows 10 (Intel Celeron N2940, 1.83 GHz, x64), using Yalmip (Löfberg, 2004) and SeDuMi (Sturm, 1999).

3.3.1 Example 1

Our goal in this experiment is to design state-feedback gains for different values of the filtering tax β, and illustrate its impact on both filtering parameter and control law. Consider the inverted pendulum system investigated in Delibaş *et al.* (2013) where the pendulum varies between $-30°$ and $+30°$ and the control signal is constrained by the voltage applied to the armature motor which cover a range from -13V up to $+13$V. Such a system is described by (3.1), where the polytope vertices are given by matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -15.148 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2.2772 \\ 5.2470 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -16.664 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$B_{w1} = \begin{bmatrix} 0 \\ 0 \\ -1.2497 \\ -2 \end{bmatrix}, \; B_{w2} = \begin{bmatrix} 0 \\ 0 \\ -1.3748 \\ -2 \end{bmatrix},$$

$D_w = 0$. Consider that the system input, which is the force applied by a DC servomotor, saturates at $|\rho| = 13$.

With the obtained designs, we simulated the system with the same noise affecting the measured parameter α. Figure 2.1 shows the influence of β on the time-varying parameter α_1, where great values of β clearly lead to smooth behavior of parameter ξ used in the control law. Therefore, the designer can tune β accordingly with the noise level in the measured parameter, mitigating unnecessary oscillations in the control signal and reducing maintenance costs. Such an aspect can be noted by the gains variation as shown in Figure 3.1. The design is performed by Theorem 3.1 with $\gamma = 1$, $x(0) = 0$ and $\delta = \mu = 1$. It is clear from Figure 3.1 that the more the parameter is filtered, the more the state-feedback gains have a smooth behavior.

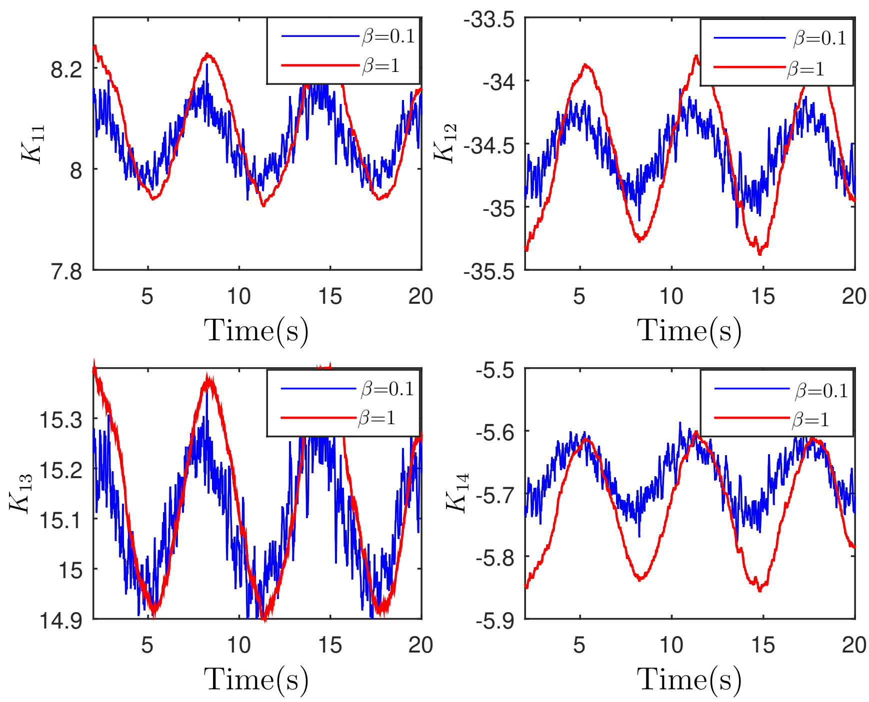

Figure 3.1: Parameter-dependent state-feedback gains behavior for different filtering values β.

The designed controller when $\beta = 0.1$ is described as

$$K(\xi(t)) = (\xi_1(t)Z_1 + \xi_2(t)Z_2)(\xi_1(t)P_1 + \xi_2(t)P_2)^{-1},$$

where

$$Z_1 = [-5.9013 \quad -0.6987 \quad 24.0984 \quad 25.1250],$$

$$Z_2 = [-5.6021 \quad -0.8114 \quad 21.3393 \quad 23.9823],$$

$$P_1 = \begin{bmatrix} 0.5720 & -0.0141 & -0.7852 & -0.1236 \\ -0.0141 & 0.3195 & 0.3480 & -0.9176 \\ -0.7852 & 0.3480 & 4.0292 & 3.1256 \\ -0.1236 & -0.9176 & 3.1256 & 9.1976 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.5747 & -0.0174 & -0.7504 & -0.1304 \\ -0.0174 & 0.3117 & 0.3258 & -0.9134 \\ -0.7504 & 0.3258 & 3.6052 & 2.9644 \\ -0.1304 & -0.9134 & 2.9644 & 9.2166 \end{bmatrix}.$$

In this example, it is discussed the influence of the filtering tax β on both time-varying parameter and state-feedback gains. It is shown that even when the time-varying parameter derivative is unbounded, LPV controllers can be calculated. Besides, the smoothest behavior of the state-feedback gains is obtained for higher values of β.

3.3.2 Example 2

Consider the following system, adapted from Montagner *et al.* (2005), that represents a helicopter in a vertical flight subject to airspeed changes, given by

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & a_{32}(t) & -0.7070 & a_{34}(t) \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ b_{12}(t) & -7.5922 \\ -5.5200 & 4.9900 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix},$$

with $B_w = \mathbf{I}_4$, $D_w = \mathbf{0}$, $-0.1134 \leq a_{32}(t) \leq 0.6844$, $-0.8667 \leq a_{34}(t) \leq 3.5125$ and $-0.7178 \leq b_{12}(t) \leq 6.8072$. The system has three time-varying parameters, hence a polytope of $N = 8$ vertices will be used to describe it.

Because we have three interconnected variables, γ, δ, and ρ, the optimization procedure $\mathcal{T}_2$ can be exploited in different ways. In the sequel, we present two sets of tests illustrating possible uses of it. In all cases, the optimized values are investigated as a function of the actuator's saturation level, and the initial condition is supposed null ($x(0) = 0$). Such an approach allows, for instance, the correct specification of an actuator to obtain the desired performance. In the first set, we use $\mathcal{T}_2$ with $f = \gamma$ and a given disturbance energy, δ^{-1}, to seek for the minimal $\mathcal{L}_2$-gain, $\sqrt{\gamma}$. In the second set of tests, we use $\mathcal{T}_2$ with $f = \mu$ and a given $\mathcal{L}_2$-gain, $\sqrt{\gamma}$, to maximize the tolerable energy disturbance, δ^{-1}. The achieved results are shown in Tables 3.1 and 3.2 and Figure 3.2, where the left-hand side ordinate axis refers to the minimal $\mathcal{L}_2$-gain achieved (green lines) and the right-hand ordinate axis refers to the maximal tolerable energy allowed (blue lines). The solid lines correspond to better achievements.

Concerning the $\mathcal{L}_2$-gain, green lines in Figure 3.2, it is clear that smaller values of γ are obtained when ρ increases, thanks to more control signal available to reject the input disturbance (see the left-hand side ordinate axis). Moreover, the $\mathcal{L}_2$-gain increases as the disturbance energy augments: the values of the solid green line, $\delta^{-1} = 0.2$, are more significant than those of the dashed green line, $\delta^{-1} = 0.1$, by 5% to 10%, approximately.

On the other hand, the maximization of the tolerable energy for a given $\mathcal{L}_2$-gain, right-hand ordinate axis, and blue lines shows that the greater the control signal, the bigger the tolerable energy is. Furthermore, more significant $\mathcal{L}_2$-gain allows more disturbance energy, as we can compare the dashed blue line ($\gamma = 5$) achieves smaller values of tolerable energy than those from the solid blue line ($\gamma = 6$).

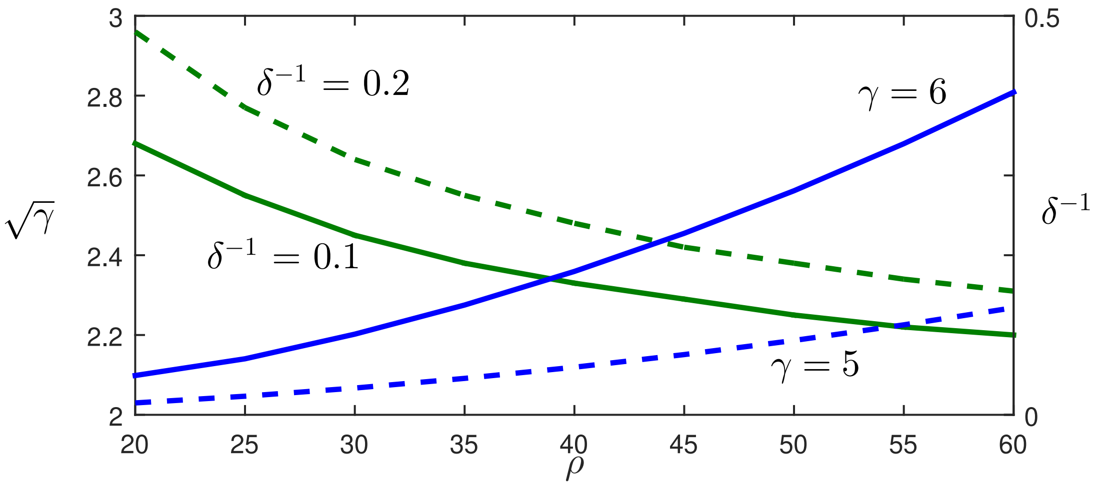

Figure 3.2: Behavior of the $\mathcal{L}_2$-gain and maximum tolerable energy δ^{-1} as a function of the saturation limit.

This example discusses the relation that exists between the saturation level ρ, the $\mathcal{L}_2$ gain γ and disturbance energy bound δ^{-1}. With the results presented, it is clear that:

1. For a fixed disturbance energy bound (δ^{-1}), bigger values of saturation level (ρ) increase the closed-loop system capability of mitigating disturbances, that is, smaller

Table 3.1: Minimal $\mathcal{L}_2$-gain for different values of saturation ρ.

ρ	$\delta^{-1} = 0.1$ $\sqrt{\gamma}$	$\delta^{-1} = 0.2$ $\sqrt{\gamma}$
20	2.68	2.96
25	2.55	2.77
30	2.45	2.64
35	2.38	2.55
40	2.33	2.48
45	2.29	2.42
50	2.25	2.38
55	2.22	2.34
60	2.20	2.31

Table 3.2: Maximum disturbance tolerance δ^{-1} for different values of saturation ρ.

ρ	$\gamma = 5$ δ^{-1}	$\gamma = 6$ δ^{-1}
20	0.014900	0.049250
25	0.023296	0.070197
30	0.033540	0.101080
35	0.045660	0.137590
40	0.059630	0.179700
45	0.075478	0.227440
50	0.093180	0.280780
55	0.112750	0.339750
60	0.134180	0.404310

values of $\mathcal{L}_2$ gain (γ) are obtained.

2. Fixing the saturation level (ρ), disturbances with more energy (δ^{-1}) lead to bigger values of γ.

3. The disturbance tolerance is higher when more control signal is available. Besides, if $\mathcal{L}_2$ gain (γ) grows, so does the disturbance energy bound (δ^{-1}) such that the systems is ISS.

3.3.3 Example 3

Consider the following system, borrowed from Karimi *et al.* (2005), with two time-varying parameters described by

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -(1+\theta_1) & -(1+\theta_2) \end{bmatrix}, \; B(\theta) = \begin{bmatrix} 0 \\ -1+\theta_2 \end{bmatrix},$$

where $-0.04 \leq \theta_1 \leq 5$, and $-5 \leq \theta_2 \leq 0.04$. In this scenario, there is no disturbance ($w = 0$) and our goal is, by using optimization procedure $\mathcal{T}_1$, design state-feedback controllers that maximizes the estimations of the region of attraction for different values of control signal saturation ρ. Consider that $\beta = 0.1$, $\mu = 1$ and $\gamma = 0.5$. In Figure 3.3, it is possible to see that the regions of attraction get bigger when more control energy is available. Response to initial conditions are shown and it is important to notice that the closed-loop system is asymptotically stable when the initial conditions belong to the region of attraction. For the case $\rho = 5$, it is also possible to see trajectories beginning outside the estimation of the region of attraction that make the closed-loop system unstable. This fact highlights that the obtained estimation of the region of attraction could be improved, however, it also provides a good measurement of the real region of attraction of the system.

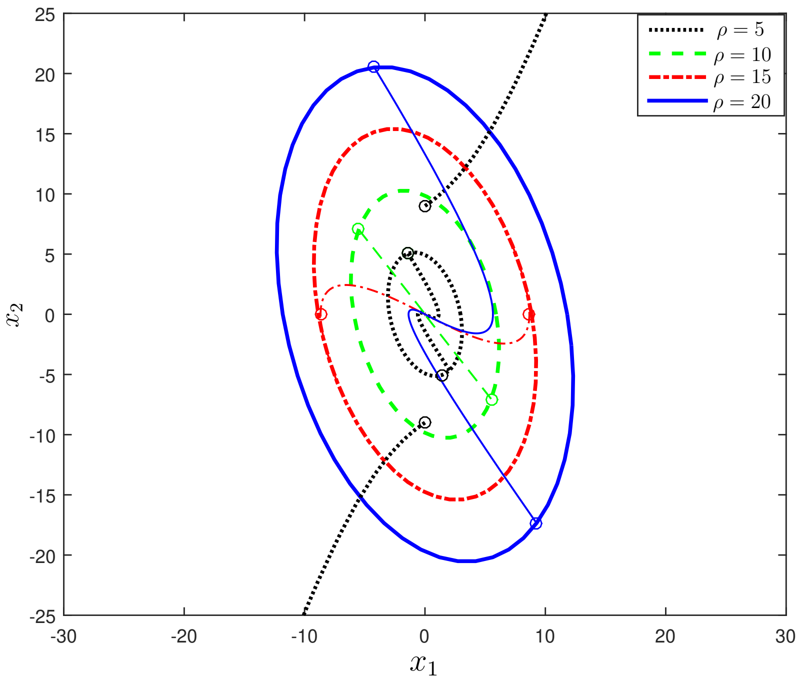

Figure 3.3: Estimated regions of attraction for different values of saturation ρ and closed-loop system response to initial conditions.

In this example, it is discussed the relation between the estimation of the region of attraction and the saturation level ρ. As expected, when more control signal is available, the region of attraction gets bigger.

3.3.4 Example 4

Consider the nonlinear plant borrowed from Figueiredo *et al.* (2020), where a discrete-time version of this model is addressed. The system consists of two tanks coupled, where one of them has a nonlinear solid inside. The input of the system is the pump power $0\% \leq u \leq 100\%$, which affects q_{in} and the output is the level (meters) in TQ-01, shown in Figure 3.4 borrowed from Figueiredo (2020).

For $0.28m \leq h_1 \leq 0.48m$, the system can be written as a polytope, whose vertices are

$$A_1 = \begin{bmatrix} -0.0111 & 0.0111 \\ 0.0155 & -0.0193 \end{bmatrix}, A_2 = \begin{bmatrix} -0.0111 & 0.0111 \\ 0.0238 & -0.0282 \end{bmatrix},$$

$$B_{u1} = B_{u2} = \begin{bmatrix} 0.5432 \times 10^{-4} \\ 0 \end{bmatrix},$$

$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D_1 = D_2 = 0$, and matrices of disturbance are

$$B_{w1} = \begin{bmatrix} 0.5432 \times 10^{-3} \\ 0 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 0.7432 \times 10^{-3} \\ 0 \end{bmatrix}.$$

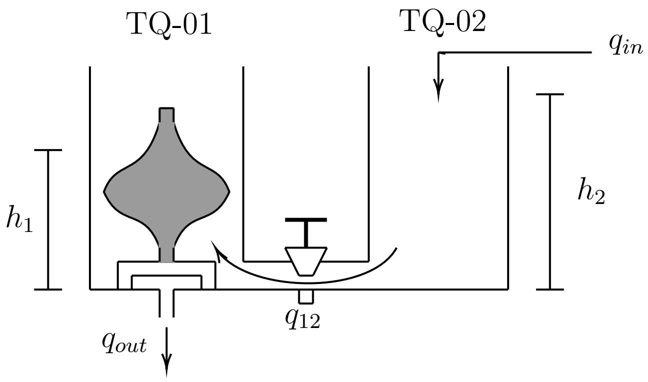

Figure 3.4: Diagram of the nonlinear coupled tanks system. From Figueiredo, 2020, p. 62.

Similarly to Figueiredo *et al.* (2020), we use an augmented version of the controlled systems to include both the state feedback control action and an integrator over the tracking error, to ensure perfect tracking under piecewise constant references. The employed topology is presented in Figure 3.5.

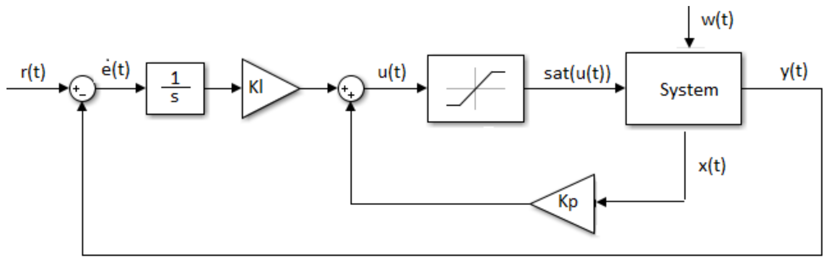

Figure 3.5: Block diagram of the closed-loop system with integrator, saturation and exogenous input.

The augmented system, due to the added integrator, is described by

$$\begin{aligned} \dot{x} &= \mathcal{A}(\alpha)x + \mathcal{B}(\alpha)\mathtt{sat}\,(u) + \mathcal{B}_w(\alpha)w \\ y &= \mathcal{C}(\alpha)x + D_w(\alpha)w \end{aligned} \tag{3.23}$$

where

$$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ -C(\alpha) & 0 \end{bmatrix}, \mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix},$$

$$\mathcal{B}_w(\alpha) = \begin{bmatrix} B_w(\alpha) \\ 0 \end{bmatrix}, \mathcal{C}(\alpha) = \begin{bmatrix} C(\alpha) & 0 \end{bmatrix}$$

and the control law is given by $u(\xi) = K(\xi)\tilde{x}$, where $K(\xi) = [K_p(\xi) \quad K_l(\xi)]$ and $\tilde{x} = [x \quad e]^\top$.

Our goal is designing, by using Theorem 3.1, state-feedback gains for system (3.23) when the $\mathcal{L}_2$-gain between the disturbance w and the output y is $\gamma = 5$. Consider that $\beta = 0.1$, $\mu = 0.1$, and $x(0) = 0$ for $\rho = 40$ and $\rho = 60$.

Figure 3.6 shows the closed-loop system step response for two different levels of control signal saturation. The input reference is changed $0.28m$ and $0.30m$ when time $t = 100s$ and $t = 1800s$, respectively. As one may see, when more control signal is available ($\rho = 60$), the system output reaches the input reference with a smaller setting time. It is important to emphasize that, because of the integrator, there is no steady-state error in both cases.

Figure 3.6 also shows the closed-loop system disturbance rejection. From $t = 3200s$ to $t = 3450s$, a disturbance of amplitude $w = 0.2$, leading to $\delta^{-1} = 10$, is applied on the system. Both controllers reject the disturbance and drive the system to the input reference with an $\mathcal{L}_2$ gain $\gamma \leq 5$, as proposed.

To simulate the experimental plant, it is considered an accuracy of $0.5cm$ on both states measurement. Hence, an uniformly distributed random signal $\kappa \in [-0.5, 0.5]$ is added to each x_1 and x_2.

In this example, it is shown an extension of the proposed conditions for the case of reference tracking. Hence, the designed controller is capable of following piecewise constant references and rejecting exogenous disturbances. State-feedback gains for a real plant model of two coupled tanks were designed. With the simulations presented, it is possible to certify the effectiveness of the controller, since the closed-loop system is capable of tracking the reference, rejecting the disturbance and when subject to initial conditions, the states of the closed-loop system converged to the origin.

The set point signal can not be chosen arbitrarily, otherwise the system may become unstable. In this work, the input reference was used after checking that the system's trajectories remains inside $\mathcal{R}_\mathcal{A}$. In Lopes *et al.* (2018), an estimate of the maximal amplitude variation of the reference signal such that the system's trajectories do not leave $\mathcal{R}_\mathcal{A}$ is presented.

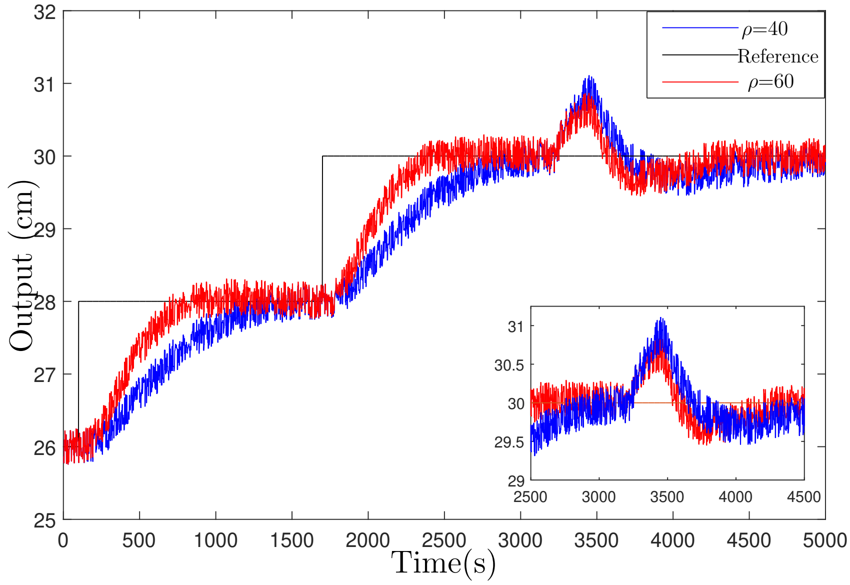

Figure 3.6: Closed-loop system input reference tracking and disturbance rejection.

3.4 Final Considerations

This chapter proposed new conditions to ISS stabilization for a class of systems that depend on time-varying parameters that may vary arbitrarily fast and present discontinuities in its behavior. Moreover, the system is subject to actuator saturation, making it nonlinear. The state-feedback controller makes use of a smoothed approximation of the real time-varying parameter (the filtered parameter ξ), reducing actuator stress. The controller depends rationally on the parameter ξ. It has been shown that higher values of ρ have provided larger estimations for the region of attraction and smaller $\mathcal{L}_2$-gain between the output and input disturbance. Furthermore, the proposed method can be used to tackle LPV systems under constraints, enhancing the robustness associated with the state-feedback controllers. Besides, the conditions were extended to the problem of designing LPV state-feedback controllers, such that the closed-loop system output is able to follow input step references.

Chapter 4

State-feedback control for continuous-time LPV systems with polynomial vector fields

This chapter is concerned with the design of state-feedback controllers for Linear Parameter Varying (LPV) polynomial continuous-time systems. The vector field presents polynomial dependence on the states. Two synthesis conditions are proposed, the first one considers arbitrary rates of variation in the time-varying parameters, while the second one provides LPV controllers that are constructed based on a smoothed approximation of the time-varying parameter. The sum of squares matrix decomposition is employed to solve the proposed conditions. The $\mathcal{L}_2$ gain is also considered to give a robustness measure of the proposed controllers. The results are illustrated with examples from the literature.

4.1 Problem formulation

Consider the following polynomial LPV system

$$\dot{x} = A(\alpha(t),x(t))x(t) + B(\alpha(t),x(t))u(t) \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input and $\alpha(t)$ is the vector of time-varying parameters belonging to the unit simplex. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n_u}$, depend polynomially on the state vector x and linearly on $\alpha(t)$. The matrices in (4.1) can be generically represented as (to simplify the notation, the dependence on t will be omitted for the time-varying parameters and states)

$$W(\alpha,x) = \sum_{i=1}^{N} \alpha_i W_i(x), \quad \alpha \in \Lambda_N, \tag{4.2}$$

where $W_i(x)$, $i = 1, \ldots, N$, are the vertices of the polytope and Λ_N is the unit simplex:

$$\Lambda_N = \big\{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\, \alpha_i \geq 0,\, i = 1, \ldots, N\big\}. \tag{4.3}$$

The goal is to provide a state-feedback control law $u = K(\alpha,x)x$ such that the closed-loop system

$$\dot{x} = \tilde{A}(\alpha,x)x, \tag{4.4}$$

with $\tilde{A} = A(\alpha,x) + B(\alpha,x)K(\alpha,x)$ is asymptotically stable.

4.2 Main Results

4.2.1 Polynomial state-feedback control law

The first result is based on the existence of a polynomial matrix $K(x)$ that assures the stability of the closed-loop system (4.4). This is the case to be considered when there is no information about the time-varying parameter trajectory α. $\upsilon = 10^{-4}$.

Theorem 4.1 *If there exist a constant matrix $P = P^T > 0$, and a polynomial matrix $Z(x)$ such that*

$$-\,\mathtt{He}(A_i(x)P + B_i(x)Z(x)) - \upsilon I \in \Sigma\left[x\right], \tag{4.5}$$

$i = 1,\ldots,N$, then the LPV polynomial system $\dot{x} = \tilde{A}(\alpha,x)x$, with $\tilde{A} = A(\alpha,x) + B(\alpha,x)K(x)$ is asymptotically stable and the polynomial state-feedback controller is given by

$$K(x) = Z(x)P^{-1}.$$

Proof: By replacing $Z(x) = K(x)P$ in (4.5), one can write

$$-\,\mathtt{He}([A_i(x) + B_i(x)K(x)]P) - \upsilon I \in \Sigma\left[x\right], \tag{4.6}$$

$i = 1,\ldots,N$. Multiplying (4.6) by α_i, $i = 1,\ldots,N$, and summing up, one has

$$-\mathtt{He}([A(\alpha,x) + B(\alpha,x)K(x)]P) - \upsilon I \in \Sigma\left[x\right],$$

which leads to

$$\tilde{A}(\alpha,x)P + P\tilde{A}(\alpha,x)^\top < \mathbf{0}. \tag{4.7}$$

Multiplying (4.7) on the left by $x^T P^{-1}$, and on the right by its transpose, one has

$$x^\top P^{-1}\tilde{A}(\alpha,x)x + x^\top \tilde{A}(\alpha,x)^\top P^{-1}x < 0. \tag{4.8}$$

Replacing (4.4) in (4.8) yields

$$\dot{x}^\top P^{-1}x + x^\top P^{-1}\dot{x} < 0,$$

or equivalently, $\dot{V}(x) < 0$ with

$$V(x) = x^\top P^{-1}x.$$

Moreover, note that $V(x)$ is positive definite, since $P > 0$ in Theorem 4.1. ∎

4.2.2 LPV Polynomial state-feedback control law

In this case, the search is for an LPV polynomial control law $u = K(\xi,x)x$, where ξ is a filtered time-varying parameter.

Theorem 4.2 *If there exist matrices $P_i = P_i^\top > 0$, $i = 1,\ldots,N$, and polynomial matrices $Z_i(x)$, $i = 1,\ldots,N$, such that*

$$-\mathtt{He}\left(A_j(x)P_i + B_j(x)Z_i(x)\right) + \tfrac{1}{\beta}(P_j - P_k) - \upsilon I \in \Sigma\left[x\right], \tag{4.9}$$

$i,j,k = 1,\ldots,N$, then the LPV polynomial system $\dot{x} = \tilde{A}(\alpha,x)x$, with $\tilde{A} = A(\alpha,x) + B(\alpha,x)K(\alpha,x)$ is asymptotically stable and the LPV polynomial state-feedback controller is given by

$$K(\xi,x) = Z(\xi,x)P(\xi)^{-1}.$$

.

Proof: Multiplying (4.9) by ξ_i, $i = 1,\ldots,N$, summing up and replacing $Z(\xi,x) = K(\xi,x)P(\xi)$ one has

$$-\mathtt{He}\left(A_j(x)P(\xi) + B_j(x)K(\xi,x)P(\xi)\right) + \frac{1}{\beta}(P_j - P_k) - \upsilon I \in \Sigma\left[x\right]. \tag{4.10}$$

Multiplying (4.10) by α_j, $j = 1,\ldots,N$, summing up and applying the same procedure with ξ_k, $k = 1,\ldots,N$, and summing up, yields

$$-\mathtt{He}([A(\alpha,x) + B(\alpha,x)K(\xi,x)]P(\xi)) + \frac{1}{\beta}(P(\alpha) - P(\alpha(t-\beta))) > \mathbf{0}. \tag{4.11}$$

By employing (2.14), it is possible to write

$$\mathtt{He}([A(\alpha,x) + B(\alpha,x)K(\xi,x)]P(\xi)) - \dot{P}(\xi) < \mathbf{0},$$

which leads to

$$\tilde{A}(\alpha,x)P(\xi) + P(\xi)\tilde{A}(\alpha,x)^\top - \dot{P}(\xi) < 0. \tag{4.12}$$

Multiplying (4.12) on the left by $x^T P(\xi)^{-1}$ and on the right by its transpose, and using the fact that $\dot{P}(\xi) = -P(\xi)\dot{P}(\xi)^{-1}P(\xi)$, one has

$$x^\top \left(P(\xi)^{-1}\tilde{A}(\alpha,x) + \tilde{A}(\alpha,x)^\top P(\xi)^{-1} + \dot{P}(\xi)^{-1}\right) x < 0, \tag{4.13}$$

Replacing (4.4) in (4.13) yields

$$\dot{x}^\top P(\xi)^{-1}x + x^\top P(\xi)^{-1}\dot{x} + x^\top \dot{P}(\xi)^{-1}x < 0,$$

or equivalently, $\dot{V}(x) < 0$ with $V(x) = x^\top P(\xi)^{-1}x$. Moreover, note that $V(x)$ is positive definite, since $P > 0$ in Theorem 4.2. ∎

4.2.3 $\mathcal{L}_2$-gain

To evaluate the $\mathcal{L}_2$-gain associated with the state-feedback controller, consider the following polynomial LPV system

$$\begin{aligned} \dot{x} &= \tilde{A}(\alpha,x)x + B_w(\alpha,x)w \\ y &= C(\alpha,x)x + D_w(\alpha,x)w \end{aligned} \tag{4.14}$$

where $w \in \mathbb{R}^{n_w}$ is the input disturbance, $y \in \mathbb{R}^{n_y}$ is the measured output and $\tilde{A}(\alpha,x)$ is the closed-loop matrix from (4.4).

Theorem 4.3 *If there exist a constant matrix $P = P^\top > 0$ and a polynomial matrix $Z(x)$ such that*

$$-M_i - \upsilon I \in \Sigma\left[x\right], \tag{4.15}$$

$i = 1,\ldots,N$*, with*

$$M_i = \begin{bmatrix} \mathtt{He}(A_i(x)P + B_i(x)Z(x)) & B_{wi}(x) & PC_i^\top(x) \\ B_{wi}^\top(x) & -\gamma^2 I & D_{wi}^\top(x) \\ C_i(x)P & D_{wi}(x) & -I \end{bmatrix}, \tag{4.16}$$

then, the closed-loop system (4.14) is asymptotically stable with a bound to the $\mathcal{L}_2$-gain given by γ. Moreover, the state-feedback control gain is given by

$$K(x) = Z(x)P^{-1}.$$

Proof: By replacing $Z(x) = K(x)P$ in (4.16), yields

$$M_i = \begin{bmatrix} \mathtt{He}([A_i(x) + B_i(x)K(x)]P) & B_{wi}(x) & PC_i^\top(x) \\ B_{wi}^\top(x) & -\gamma^2 I & D_{wi}^\top(x) \\ C_i(x)P & D_{wi}(x) & -I \end{bmatrix},$$

Multiplying (4.15) by α_i, $i = 1,\ldots,N$, and summing up, one can write

$$\begin{bmatrix} \mathtt{He}(\tilde{A}(\alpha,x)P) & B_w(\alpha,x) & PC^\top(\alpha,x) \\ B_w{}^\top(\alpha,x) & -\gamma^2 I & D_w{}^\top(\alpha,x) \\ C(\alpha,x)P & D_w(\alpha,x) & -I \end{bmatrix} < \mathbf{0}.$$

By applying a congruence transformation with

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

yields

$$\begin{bmatrix} \mathtt{He}(P^{-1}\tilde{A}(\alpha,x)) & B_w(\alpha,x) & PC^T(\alpha,x) \\ B_w{}^\top(\alpha,x) & -\gamma^2 I & D_w{}^\top(\alpha,x) \\ C(\alpha,x)P & D_w(\alpha,x) & -I \end{bmatrix} < \mathbf{0}.$$

By means of Schur Complement and considering the system described as in (4.14), one can compute the $\mathcal{L}_2$-gain condition as

$$\dot{x}^\top P^{-1}x + x^\top P^{-1}\dot{x} + y^\top y - \gamma^2 w^\top w < 0,$$

where the Lyapunov function is $V(x) = x^\top P^{-1}x > 0$. $\blacksquare$

In the sequel the $\mathcal{L}_2$ gain is established by using the filtered Lyapunov function.

Theorem 4.4 *If there exist matrices $P_i = P_i^\top > 0$ and $Z_i(x)$ such that*

$$-W_{i,j,k} - \upsilon I \in \Sigma\left[x\right],$$

$i,j,k = 1,\dots,N$, with

$$W_{i,j,k} = \begin{bmatrix} \mathcal{T} & B_{wj}(x) & P_iC_j(x)^\top \\ B_{wj}^\top(x) & -\gamma^2 I & D_{wj}^\top(x) \\ C_j(x)P_i & D_{wj}(x) & -I \end{bmatrix} \tag{4.17}$$

and

$$\mathcal{T} = \mathtt{He}(A_j(x)P_i + B_j(x)Z_i(x)) - \frac{1}{\beta}(P_j - P_k),$$

then, the closed-loop system (4.14) is asymptotically stable with a bound to the $\mathcal{L}_2$-gain given by γ. Moreover, the state-feedback control gain is given by

$$K(\xi,x) = Z(\xi,x)P(\xi)^{-1}.$$

Proof: Multiplying (4.17) by ξ_i, $i = 1,\dots,N$, summing up and replacing $Z(\xi,x) = K(\xi,x)P(\xi)$ one has

$$W_{j,k} = \begin{bmatrix} \mathcal{T} & B_{wj}(x) & P(\xi)C_j(x)^\top \\ B_{wj}^\top(x) & -\gamma^2 I & D_{wj}^\top(x) \\ C_j(x)P(\xi) & D_{wj}(x) & -I \end{bmatrix} < \mathbf{0} \tag{4.18}$$

with

$$\mathcal{T} = \mathtt{He}(A_j(x) + B_j(x)K(\xi,x)P(\xi)) - \frac{1}{\beta}(P_j - P_k).$$

Multiplying (4.18) by α_j, $j = 1,\dots,N$, summing up and applying the same procedure with ξ_k, $k = 1,\dots,N$, and summing up, yields

$$\begin{bmatrix} \mathcal{Q} & B_w(\alpha,x) & P(\xi)C^\top(\alpha,x) \\ {B_w}^\top(\alpha,x) & -\gamma^2 I & {D_w}^\top(\alpha,x) \\ C(\alpha,x)P(\xi) & D_w(\alpha,x) & -I \end{bmatrix} < \mathbf{0},$$

with $\mathcal{Q} = \tilde{A}(\alpha,x)P(\xi) + P(\xi)\tilde{A}(\alpha,x)^\top - \dot{P}(\xi)$. By applying a congruence transformation with

$$\begin{bmatrix} P(\xi)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and using $\dot{P}(\xi) = -P(\xi)\dot{P}(\xi)^{-1}P(\xi)$ it is possible to write

$$\begin{bmatrix} \mathcal{R} & P(\xi)^{-1}B_w(\alpha,x) & C^\top(\alpha,x) \\ {B_w}^\top(\alpha,x)P(\xi)^{-1} & -\gamma^2 I & {D_w}^\top(\alpha,x) \\ C(\alpha,x) & D_w(\alpha,x) & -I \end{bmatrix} < \mathbf{0},$$

where $\mathcal{R} = P(\xi)^{-1}\tilde{A}(\alpha,x) + \tilde{A}(\alpha,x)^\top P(\xi)^{-1} + \dot{P}(\xi)^{-1}$. By means of the Schur complement, it is possible to get the $\mathcal{L}_2$-gain condition as

$$\dot{V}(x) + y^\top y - \gamma^2 w^\top w < 0, \tag{4.19}$$

with $V(x) = x^\top P(\xi)^{-1}x$. Note that $P(\xi)$ is positive definite in Theorem 4.4, concluding the proof. ∎

4.3 Numerical Examples

This section illustrates the results for the design of stabilizing controllers for LPV polynomial continuous-time systems. The effects of the filtered time-varying parameter $\xi(t)$ in the control design will be explored. The routines were implemented in MATLAB, version 8.2.0.701 (R2013b) using SOSTOOLS (Papachristodoulou *et al.*, 2013) and SeDuMi (Sturm, 1999).

4.3.1 Example 1

Consider the following system, adapted from Prajna *et al.* (2004),

$$\dot{x} = \begin{bmatrix} -x_1^2 + x_1 & 1 \\ \theta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad 0 \leq \theta \leq 4, \tag{4.20}$$

where θ is the varying parameter, but it is not available online. By using Theorem 4.1, the state-feedback controller designed to guarantee the asymptotic stability of the closed-loop system is $K(x) = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, where

$k_1 = -2.964x_1^2 + 0.9868x_1 - 7.905,$

$k_2 = -1.586x_1^2 + 0.2698x_1 - 3.122.$

Consider now that the parameter is available, hence it can be used to compute the state-feedback gain, as well its smoothed approximation. The controller designed with Theorem 4.2 is

$$K(\xi,x) = (\xi_1 Z_1(x) + \xi_2 Z_2(x))(\xi_1 P_1 + \xi_2 P_2)^{-1},$$

where $Z_1(x) = [z_a \quad z_b]$ and $Z_2(x) = [z_c \quad z_d]$ with

$z_a = -0.3275x_1^2 + 0.2408x_1 - 1.185,$

$z_b = -0.4872x_1^2 - 0.03752x_1 - 0.2317,$

$z_c = -0.2838x_1^2 + 0.2049x_1 - 1.111,$

$z_d = -0.4898x_1^2 - 0.05427x_1 - 0.2652,$

$$P_1 = \begin{bmatrix} 0.2502 & -0.3412 \\ -0.3412 & 0.9918 \end{bmatrix}, P_2 = \begin{bmatrix} 0.25 & -0.2955 \\ -0.2955 & 0.8706 \end{bmatrix}.$$

The controller was designed for $\beta = 0.1$, which leads to ξ, a smoothed approximation of α. The influence of β is depicted in Figure 2.1.

As expected, the value of β also impacts on the values of the state-feedback gains. As shown in Figure 4.1, the highest β used provides the smoothest behavior for k_1 and k_2, where $K(\xi,x) = [k_1 \quad k_2]$. However, it is also possible to notice that the largest magnitude

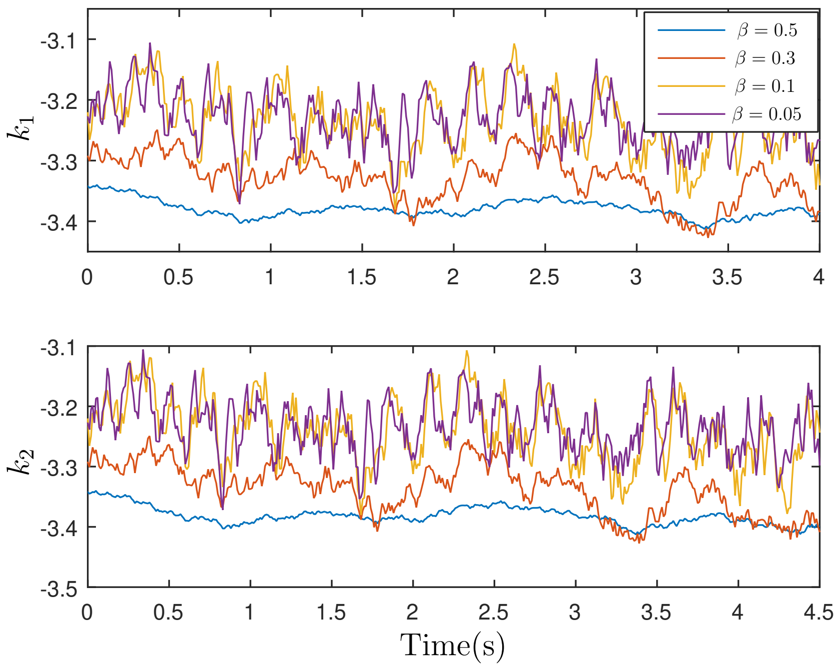

Figure 4.1: State-feedback gains variation for different β.

are assumed by k_1 and k_2 when $\beta = 0.5$. It means that more energy is used to stabilize the system. Consider that the control energy spent is computed by the function

$$\mathbb{U} = \int_0^\infty u^\top(t)u(t)dt, \tag{4.21}$$

where $u(t)$ is the control signal. The values of $\mathbb{U}$ for the employed filtering values (β) are presented in Table 4.1. It can be seen that higher values of β spent more control energy, which is expected since a smooth behavior of the control signal is obtained.

Table 4.1: Control energy for different values of β.

β	0.05	0.1	0.3	0.5
$\mathbb{U}$	1.1958	1.2493	1.5758	1.8388

4.3.2 Example 2

Consider the following polynomial LPV system, adapted from Tanaka *et al.* (2009), with matrices

$$A(\theta,x) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & -1+\theta \end{bmatrix},$$

$$B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_w = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T,\ D_w = 0,$$

$0 \leq \theta \leq 4$. The goal is designing a state-feedback controller that minimizes the $\mathcal{L}_2$-gain under two different circumstances: on the first one, the parameter is not available online and on the second the parameter is available all the time, so it can be used to design the controller. In both cases, it is desired that the stabilization works for arbitrary rates of variation on the time-varying parameter. For the first case, Theorem 4.3 should be used, the state-feedback gain is designed with a matrix $Z(x)$ with degree $[0:2]$. The $\mathcal{L}_2$-gain is given by $\gamma = 2.0133$ and 16 variables were employed to solve the problem. If θ is known through time, Theorem 4.4 can be used. The $\mathcal{L}_2$-gain is given by $\gamma = 2.0339$ and 31 variables were used. The controller was designed for $\beta = 0.05$ and matrices $Z_i(x)$ with degree $[0:2]$.

4.3.3 Example 3

Consider the nonlinear Lorenz chaotic system with polynomial dynamic, adapted from Rakhshan *et al.* (2018)

$$A(\theta,x) = \begin{bmatrix} -a & a & 0 \\ \theta & -1 & -x_1 \\ x_2 & 0 & -b \end{bmatrix},\ C(x) = \begin{bmatrix} 0.1 \\ 0 \\ x_1 + x_2 \end{bmatrix}^T$$

$$B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ B_w(\theta) = \begin{bmatrix} 1 \\ 0 \\ 0.1\theta \end{bmatrix},\ D_w = 0,\ a = 10,\ b = \tfrac{8}{3}.$$

The time-varying parameter is $\theta \in [30,k]$ and, according to Alam & Ahmed (2017), in the absence of control, the system is chaotic when $\theta \in [27.9, 99.6]$. The results, provided by Theorems 4.3 and 4.4, are presented in Table 4.2. Both approaches are formulated with matrices Z of degree $[0:4]$ on the states x_1 and x_2 and $[0]$ on x_3. One may see that when k grows, the $\mathcal{L}_2$ gain also increases. Although Theorem 4.4 design LPV polynomial stabilizing controllers it presents greater values of γ when compared with Theorem 4.3. This is because the controller provided by Theorem 4.4 depends upon a smoothed approximation of the time-varying parameter. Theorems 4.3 and 4.4 used 52 and 103 variables to solve the problem, respectively.

Table 4.2: $\mathcal{L}_2$-Gain γ when considering Theorem 4.3 and Theorem 4.4 for different values of k.

k	Theorem 4.3	Theorem 4.4 ($\beta = 0.05$)
35	24.98	27.71
40	26.00	30.62
45	28.76	32.06

By using Theorem 4.1, it is possible design a state-feedback controller considering a matrix Z with degree $g_Z = [0 : 4]$ on the states. The parameter variation and the phase portrait of the open-loop (chaotic behavior) and closed-loop (considering four different initial conditions) system are depicted in Figure 4.2.

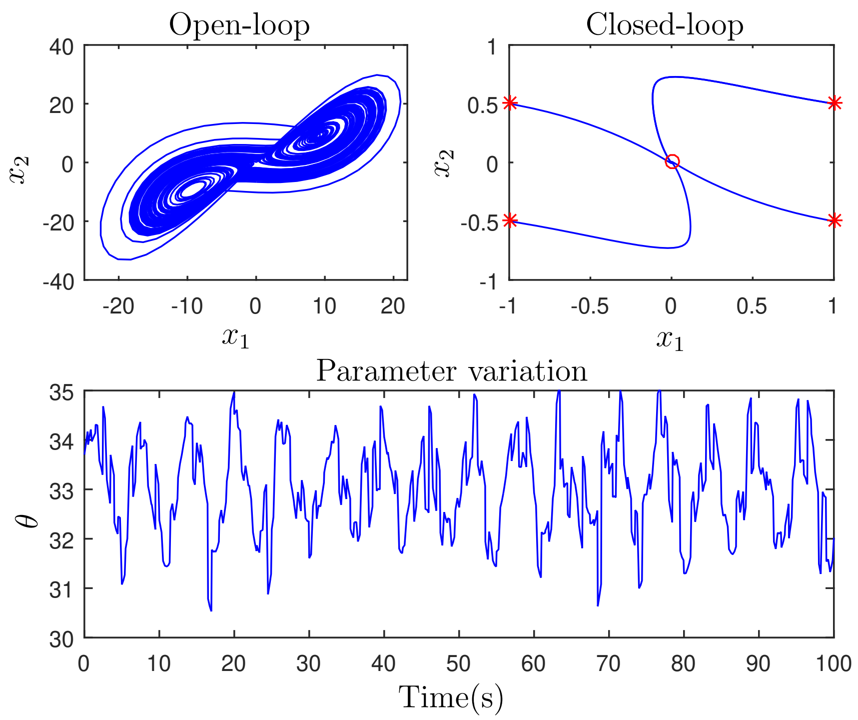

Figure 4.2: Response to initial condition and θ behavior.

4.4 Final Considerations

New conditions for the synthesis of state-feedback controllers for LPV polynomial continuous-time systems are presented. Two approaches, based on the sum of squares decomposition, have been developed: the first one is based on a fixed robust state-feedback gain, providing stabilization for arbitrary fast variations of the time-varying parameter. The second one is an LPV controller, making use of a smoothed approximation of the real time-varying parameter and providing stabilization not only for arbitrary fast variation but also for piecewise continuous behavior of the time-varying parameters. Both conditions are extended to compute controllers that minimize the $\mathcal{L}_2$-gain from the input disturbance to the output of the system. It is important to emphasize the fact that the control strategies proposed in this chapter can handle the existence of time-varying parameters and polynomial dependency in the vector field simultaneously.

Chapter 5

Polynomial systems with input constraints

This chapter presents synthesis conditions for continuous-time varying systems with state polynomial dependency and input constraints (the filtering parameter approach is not applied). Two conditions, based on sum of squares decomposition, are developed. The $\mathcal{L}_2$ gain is considered as a design requirement. An augmented version of the system, which includes an integrator over the tracking error, is presented, allowing the reference following. Numerical examples illustrate the obtained results.

5.1 Problem Formulation

Consider the following LPV system subject to saturating actuators (to simplify the notation, the dependence on t will be omitted for $\alpha(t)$, $w(t)$, $x(t)$ and $y(t)$):

$$\dot{x} = A(\alpha,x)Z(x) + B(\alpha,x)u(x) \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $Z(x) \in \mathbb{R}^N$ is the vector of monomials, $u(x) \in \mathbb{R}^{n_u}$ is the polynomial control input and α is the time-varying parameter. Matrices in (5.1) can be written in terms of its vertices as

$$Q(\alpha,x) = \sum_{i=1}^{N} \alpha_i Q_i(x), \quad \alpha \in \Lambda_N, \tag{5.2}$$

where $Q_i(x)$, $i = 1, \ldots, N$, are the polynomial state dependent vertices of the polytope and Λ_N is the unit simplex. There are no constraints on the rates of variation of the time-varying parameters that can move arbitrarily fast inside the unit simplex.

The control input is bounded in magnitude by

$$\mathcal{U} = \{u(x) \in R^{n_u} : |u_\ell(x)| \leq \rho_\ell, \ell = 1, \ldots, n_u\}. \tag{5.3}$$

Before introducing the main results, let us define the matrix L, whose elements are computed as

$$L_{i,j} = \frac{\partial Z_i(x)}{\partial x_j}, i = 1, \ldots, N, j = 1, \ldots, n.$$

Note that $\dot{Z}(x) = L\dot{x}$, moreover, when $Z(x) = x$ one has $L = I_n$.

5.2 Main Results

The first result makes use of the polynomial Lyapunov matrix to recover the state-feedback controller. As a consequence, the controller will be rational in the state-variables x, unless a constant matrix is employed.

The estimation of the region of attraction for the closed-loop system is via the level set of a polynomial Lyapunov function

$$\mathcal{R}_{\mathcal{E}}(P^{-1}) = \{x \in \mathbb{R}^n : Z(x)^\top P^{-1}(x) Z(x) \leq 1\}. \tag{5.4}$$

A rational Lyapunov function is employed, the next theorems present the results.

Theorem 5.1 *If there exists a symmetric positive definite polynomial matrix $P(x) \in \mathbb{R}^{n\times n}$, and a polynomial matrix $G(x) \in \mathbb{R}^{n_u \times n}$ such that*

$$\begin{bmatrix} P(x) & G(x)^\top e_\ell \\ e_\ell^\top G(x) & \rho^2_{(\ell)} \end{bmatrix} - \epsilon I \in \Sigma\left[x\right], \tag{5.5}$$

$$-\tau(x) - \epsilon I \in \Sigma\left[x\right] \tag{5.6}$$

hold for all $i = 1,\ldots,N$ and $k = 1,2$, where

$$\tau(x) = \mathtt{He}\big(LA_i(x)P(x) + LB_i(x)G(x)\big) - \sum_{j=1}^{n} \frac{\partial P(x)}{x_j} A_{ij}(x) Z(x) - \sum_{\ell=1}^{n_u} (-1)^k \rho_\ell \sum_{j=1}^{n} \frac{\partial P(x)}{x_j} B_{ij\ell}(x),$$

$A_{ij}(x)$ is the jth row of the matrix $A_i(x)$, $B_{ij\ell}(x)$ is the (j,ℓ) element of the matrix $B_i(x)$ and e_ℓ is the ℓth column of the identity matrix of size ℓ, then,

$$K(x) = G(x)P^{-1}(x) \tag{5.7}$$

is the polynomial control law that assures that for any initial condition $x_0 \in \mathcal{R}_{\mathcal{E}}(P^{-1})$ in (5.4)*, the signal $u(x) = K(x)Z(x)$ belongs to $\mathcal{U}$ in* (5.3)*.*

Proof: By applying the Schur complement in (5.5), one has

$$P(x) - \frac{1}{\rho^2} G(x)^\top e_\ell e_\ell^\top G(x) > \mathbf{0}. \tag{5.8}$$

By applying the change of variables $G(x) = K(x)P(x)$, and pre- and post-multiplying the inequality above by $P^{-1}(x)$, one has

$$P^{-1}(x) - \frac{1}{\rho^2} K(x)^\top e_\ell e_\ell^\top K(x) > \mathbf{0}. \tag{5.9}$$

Multiplying by $Z(x)^\top$ on the left and by $Z(x)$ on the right yields

$$Z(x)^\top P^{-1}(x) Z(x) - \frac{1}{\rho^2} Z(x)^\top K(x)^\top e_\ell e_\ell^\top K(x) Z(x) > 0, \tag{5.10}$$

$\ell = 1, \ldots, n_u$, that implies the set condition $\mathcal{R}(P^{-1}) \subset \mathcal{U}$.

Note that $-\rho_\ell \leq u \leq \rho_\ell$, in this way, evaluating (5.6) for $k = 1,2$, allows us to write

$$\mathtt{He}\big(LA_i(x)P(x) + LB_i(x)G(x)\big) - \sum_{j=1}^{n} \frac{\partial P(x)}{x_j}(A_j(x)Z(x) + Bj(x)u(x)) < \mathbf{0},$$

$i = 1, \ldots, N$. Multiplying (5.2) by $\alpha_i, i, \ldots, N$ and summing up, gives

$$\mathtt{He}\big(LA(\alpha,x)P(x) + LB(\alpha,x)G(x)\big) - \sum_{j=1}^{n} \frac{\partial P(x)}{x_j}\dot{x}_j < \mathbf{0}.$$

By applying the change of variables $G(x) = K(x)P(x)$, and pre- and post-multiplying the inequality above by $P^{-1}(x)$, one has

$$\mathtt{He}\big(P^{-1}(x)LA(\alpha,x) + P^{-1}(x)LB(\alpha,x)K(x)\big) + \dot{P}^{-1}(x) < \mathbf{0}.$$

Pre and post-multiplying (5.2) by $Z(x)^\top$ and $Z(x)$, respectively yields $\dot{V}(x) < 0$, with $V(x) = Z(x)^\top P^{-1}(x) Z(x)$. The polynomial matrix P(x) is assumed to be positive, implying that $V(x)$ is also positive definite, concluding the proof. ■

$\mathcal{L}_2$-gain

To compute the $\mathcal{L}_2$-gain associated with the state-feedback controller, consider the following polynomial LPV system

$$\begin{aligned} \dot{x} &= \tilde{A}(\alpha,x)Z(x) + B(\alpha,x)u(x) + B_w(\alpha,x)w \\ y &= C(\alpha,x)Z(x) + D_w(\alpha,x)w \end{aligned} \tag{5.11}$$

where $w \in \mathbb{R}^{n_w}$ is the input disturbance and $y \in \mathbb{R}^{n_y}$ is the measured output.

Theorem 5.2 *If there exist a symmetric positive definite polynomial matrix $P(x) \in \mathbb{R}^{n \times n}$, a polynomial matrix $G(x) \in \mathbb{R}^{n_u \times n}$ such that*

$$-\begin{bmatrix} \tau(x) & LB_{wi}(x) & PC_i^\top(x) \\ B_{wi}^\top(x)L^\top & -I & D_{wi}^\top(x) \\ C_i(x)P & D_{wi}(x) & -\gamma I \end{bmatrix} - \epsilon I \in \Sigma\left[x\right], \tag{5.12}$$

$$\begin{bmatrix} P(x) & G(x)^\top e_r \\ e_r^\top G(x) & \rho^2_{(\ell)} \end{bmatrix} - \epsilon I \in \Sigma[x], \tag{5.13}$$

hold for all $i = 1,\ldots,N$ and $k = 1,2$, then the closed-loop system (5.11) *is asymptotically stable with a bound to the $\mathcal{L}_2$-gain given by $\sqrt{\gamma}$ and a bound to the control signal amplitude given by ρ. Moreover, the state-feedback control gain is given by*

$$K(x) = Z(x)P^{-1}(x).$$

Proof: By replacing $G(x) = K(x)P(x)$ in (5.12), evaluating for $k = 1,2$, multiplying by $\alpha_i = 1,\ldots,N$ and summing up, yields

$$\begin{bmatrix} \eta(\alpha,x) & B_w(\alpha,x) & PC^\top(\alpha,x) \\ {B_w}^\top(\alpha,x) & -\gamma^2 I & {D_w}^\top(\alpha,x) \\ C(\alpha,x)P & D_w(\alpha,x) & -I \end{bmatrix} < \mathbf{0},$$

where $\eta(\alpha,x)$ is

$$\eta(\alpha,x) = \mathtt{He}\big(LA(\alpha,x)P(x) + LB(\alpha,x)G(x)\big) - \sum_{j=1}^{n} \frac{\partial P(x)}{x_j}\dot{x}_j < \mathbf{0}.$$

By applying a congruence transformation with

$$\begin{bmatrix} P^{-1}(x) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

yields

$$\begin{bmatrix} \tilde{A}^\top(\alpha,x)L^\top P^{-1}(x) + P^{-1}(x)L\tilde{A}(\alpha,x) + \dot{P}^{-1}(x) & P^{-1}(x)LB_w(\alpha,x) & C^\top(\alpha,x) \\ {B_w}^\top(\alpha,x)L^\top P^{-1}(x) & -I & {D_w}^\top(\alpha,x) \\ C(\alpha,x) & D_w(\alpha,x) & -\gamma I \end{bmatrix} < \mathbf{0}.$$

By means of Schur Complement, considering the system described as in (5.11) and a state-feedback control law $u(x) = K(x)Z(x)$, one can compute the $\mathcal{L}_2$-gain condition as

$$\dot{Z}^\top(x)P^{-1}(x)Z(x) + Z(x)^\top P^{-1}(X)\dot{Z}(x) + Z^\top(x)\dot{P}^{-1}(x)Z(x) + y^\top y - \gamma^2 w^\top w < 0,$$

where the Lyapunov function is $V(x) = Z^\top(x)P^{-1}(x)Z(x) > 0$. The proof of (5.13) follows the same steps presented in Theorem 5.1. ∎

5.2.1 Optimization design procedures

The proposed conditions work with different variables that can be optimized, leading to closed-loop systems with specific performances.

Maximization of $\mathcal{R}_\mathcal{E}$: to maximize the set of all initial conditions such that the closed-loop system is asymptotically stable, one approach is maximizing $\mathcal{E}(H,1)$, $H \in \mathbb{R}^{n\times n}$, such that $\mathcal{E}(H,1) \subset \mathcal{L}_\mathcal{V}(1)$, which can ensured by

$$\mathcal{T}_1 : \begin{cases} \displaystyle\min_{P(x),G(x)} & \texttt{trace}(H) \\ \text{subject to:} & \text{Theorems 5.1 or 5.2} \\ & \begin{bmatrix} P(x) & \mathbf{I} \\ \star & H \end{bmatrix} - \epsilon I \in \Sigma\left[x\right]. \end{cases} \tag{5.14}$$

The other optimization procedures can be proposed in the form of

$$\mathcal{T}_2(f) : \begin{cases} \min f \\ \text{subject to:} \quad \text{Theorems 5.1 or 5.2} \end{cases} \tag{5.15}$$

where the objective function f can be chosen according to:

1. Choose $f = \rho$ to search for the smallest amount of control signal necessary to stabilize the system.

2. Choose $f = \gamma$ to minimize the $\mathcal{L}_2$-gain between the disturbance w and the output y.

5.2.2 Piecewise constant reference tracking

Similarly to Chapter 3, an augmented version of the system can be formulated, so the state-feedback control action can be combined an integrator over the tracking error. Hence, it is possible to ensure a null steady-state error for piecewise constant references. The augmented system is described by

$$\begin{aligned} \dot{x} &= \mathcal{A}(\alpha,x)\tilde{Z}(x) + \mathcal{B}(\alpha,x)u(x) + \mathcal{B}_w(\alpha,x)w \\ y &= \mathcal{C}(\alpha,x)\tilde{Z}(x) + D_w(\alpha,x)w \end{aligned} \tag{5.16}$$

where

$$\mathcal{A}(\alpha,x) = \begin{bmatrix} A(\alpha,x) & 0 \\ -C(\alpha,x) & 0 \end{bmatrix}, \mathcal{B}(\alpha,x) = \begin{bmatrix} B(\alpha),x \\ 0 \end{bmatrix},$$

$$\mathcal{B}_w(\alpha,x) = \begin{bmatrix} B_w(\alpha,x) \\ 0 \end{bmatrix}, \mathcal{C}(\alpha,x) = \begin{bmatrix} C(\alpha,x) & 0 \end{bmatrix}$$

and the control law is given by $u(x) = \tilde{K}(x)\tilde{Z}(x)$, where $\tilde{K}(x) = [K_p(x) \quad K_l(x)]$ and $\tilde{Z}(x) = [x \quad e]^\top$.

5.3 Numerical Examples

5.3.1 Example 1

Consider the following example, borrowed from Jennawasin & Banjerdpongchai (2018). All experiments below use robust controllers, where the degree of matrices G and P on states is $d_x = 0$ and $L = \mathbf{I}$.

$$A = \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ -2 & \theta \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \theta \in [0,1].$$

Our goal is designing state-feedback control laws such that the control signal remains inside specific interval, that is $|u(x)| \leq \rho$, and estimating the region of attraction where the closed-loop system is asymptotically stable. Figure 5.1 shows that when more control signal is available, bigger regions of attraction are obtained.

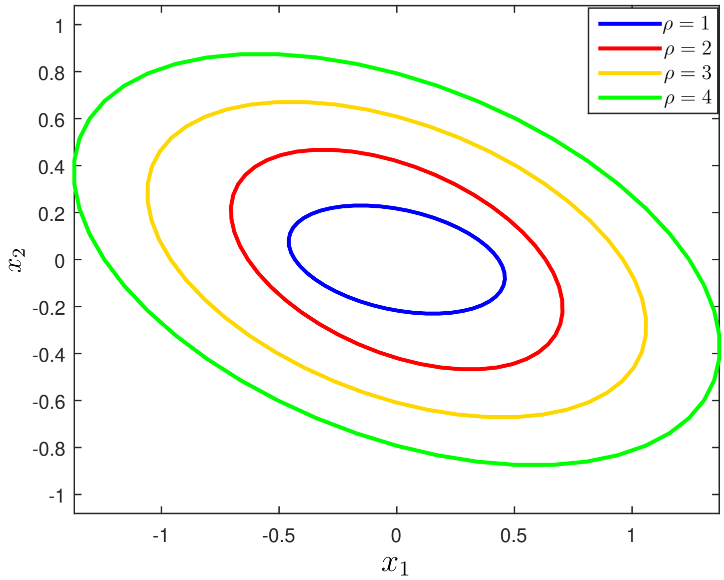

Figure 5.1: Estimated regions of attraction for different values of ρ.

In Figure 5.2, responses to initial conditions respecting each region of attraction presented above are presented. It is clear that, for all cases, the system converges to the origin and the input amplitude constraint holds.

The optimization procedure 5.14 can be used to maximize the regions of attraction for all values of ρ. The results presented in Figure 5.3 evince the growth of such ellipsoids.

Consider now that the system is subject to external disturbance. Our goal is designing state-feedback controllers by using Theorem 5.2 and optimization procedure 5.15 such that the $\mathcal{L}_2$-gain between the disturbance w and the output y is minimized. The disturbance and output matrices are, respectively

$$B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 25 & 25 \end{bmatrix}.$$

As can be seen in Table 5.1, more control energy available leads to smaller values of $\sqrt{\gamma}$, as expected.

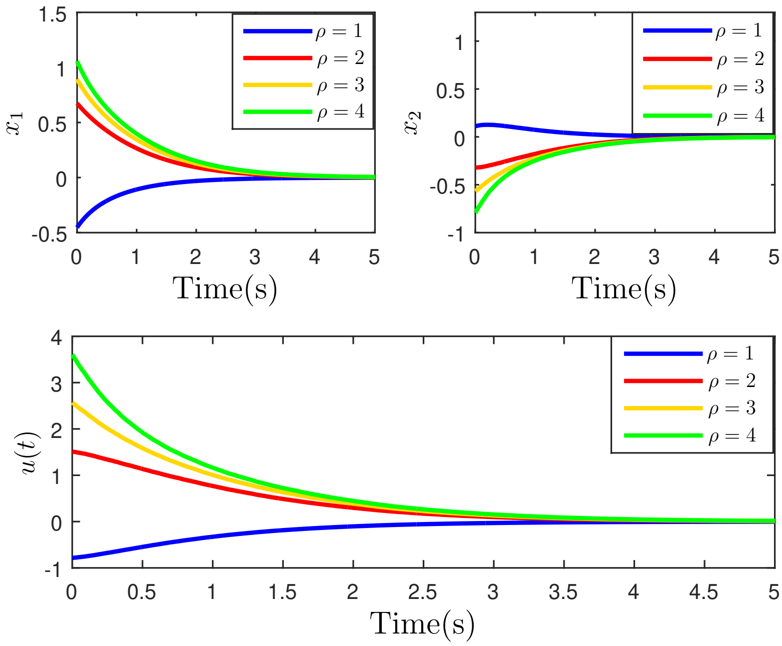

Figure 5.2: Control signals inside the limit ρ for all cases.

Table 5.1: $\mathcal{L}_2$-gain between the disturbance w and the output y.

ρ	1	2	3	4
$\sqrt{\gamma}$	0.3760	0.2247	0.1741	0.1421

5.3.2 Example 2

Consider the following example

$$A = \begin{bmatrix} 0.01(x_1+1)^2 & -1 \\ 0 & \theta \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \theta = [0{,}2].$$

With Theorem 5.1, it is not possible stabilizing this system by using a robust controller when the input constraint is $\rho = 2$. Theorem 5.1 and a matrix $P(x)$ of degree $d_x = [0{,}2]$ should be used to accomplish that. For matrix G, $d_x = 0$ was used. The state-feedback controller is defined as

$$K(x) = GP(x)^{-1},$$

where

$G = [-0.0008858, \quad -2028]$ and

$$P(x) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with,

$P_{11} = 4.693 \times 10^{-5} x_1^2 - 8.051 \times 10^{-8} x_1 x_2 + 2.804 \times 10^{-8} x_2^2 + 0.000174;$

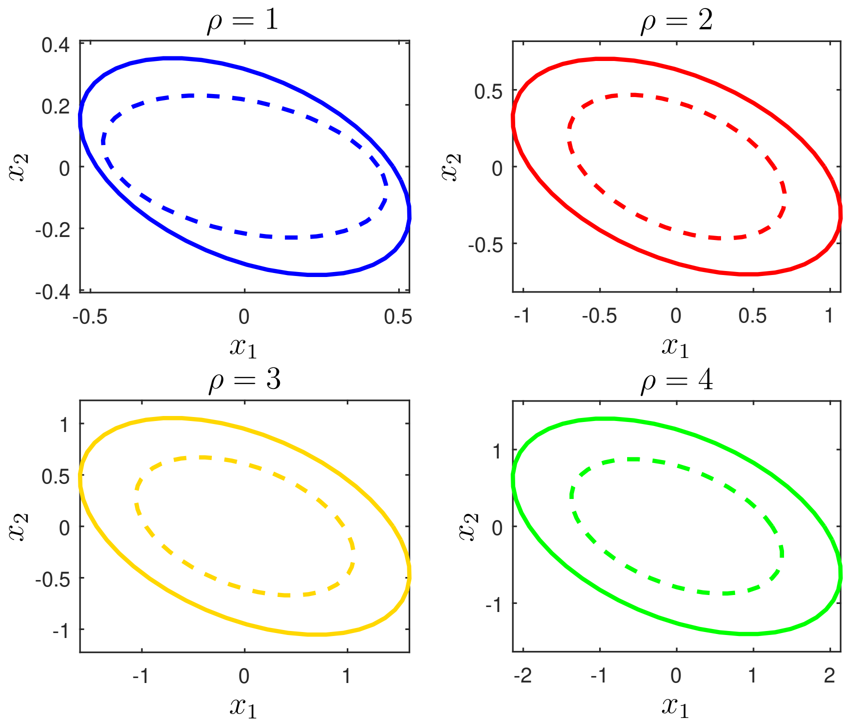

Figure 5.3: Region of attraction after optimization procedure.

$$P_{12} = 2.133 \times 10^{-6}x_1^2 + 4.611 \times 10^{-5}x_1x_2 + 3.434 \times 10^{-8}x_2^2 + 0.001546;$$
$$P_{21} = 2.133 \times 10^{-6}x_1^2 + 4.611 \times 10^{-5}x_1x_2 + 3.434 \times 10^{-8}x_2^2 + 0.001546;$$
$$P_{22} = 2.672 \times 10^{-5}x_1^2 - 3.515 \times 10^{-6}x_1x_2 + 4.629 \times 10^{-5}x_2^2 + 0.02513.$$

5.3.3 Example 3

Consider the following LPV model

$$A = \begin{bmatrix} -x_1^2 - x_2^2 - 2 + 0.1x_2 & 0 \\ x_2 & \theta \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \theta = [1, 3].$$

Our goal is designing a state-feedback controller such that the closed-loop system is capable of tracking piecewise constant references when $\rho = 50$. By using Theorem 5.1, the augmented matrices presented and setting degree $d_x = 0$ for both matrices G and P, it is possible achieving such objective. Figure 5.4 shows the closed-loop system response for step and sinusoidal references and, for both cases, the output is controlled. The control signal maximum amplitude is just 6% of the maximum available, hence the system is able to track even bigger references signals.

As discussed in chapter 3, the set point signal can not be arbitrary. The input reference was chosen after a checking that the system's trajectories do not leave $\mathcal{R}_{\mathcal{A}}$.

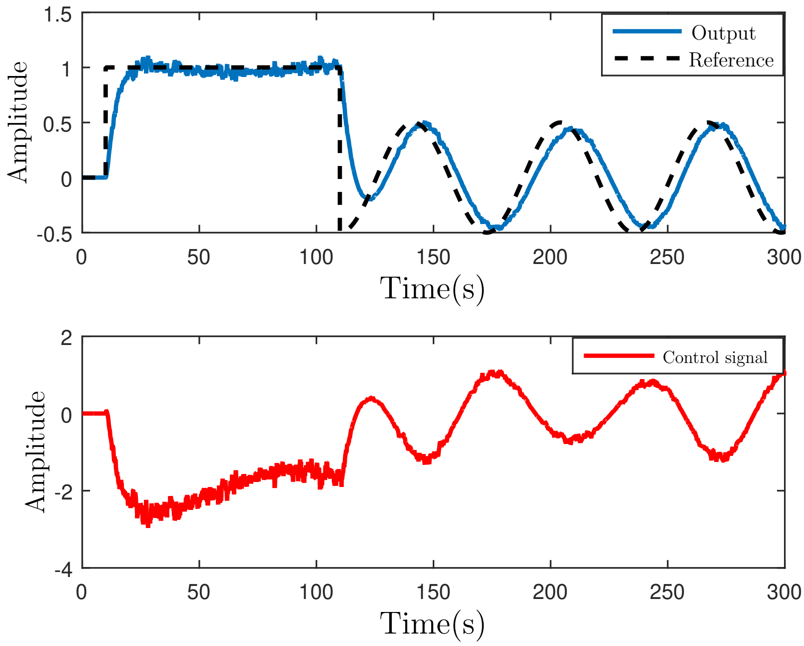

Figure 5.4: Reference tracking and control signal.

5.4 Final Considerations

New conditions for the synthesis of state-feedback controllers for continuous-time varying systems with state polynomial dependency and input constraints are presented. As a design requirement, the $\mathcal{L}_2$ gain between input disturbances and the system's output is considered. Numerical examples were used to evince the relation between the control signal amplitude available and the region of attraction, as well the $\mathcal{L}_2$ gain. The problem of reference tracking was also considered.

Chapter 6

Conclusion

In this work, conditions for design of state-feedback controllers for different kinds of continuous-time varying systems were presented: linear and state polynomial dependent ones.

In Chapter 2, the mathematical tools necessary for developing the conditions proposed in this work are presented.

In Chapter 3, by using a filtered version of the time-varying parameter, which may reduce the actuator stress, conditions for the ISS stabilization of LPV continuous-time systems with actuator saturation were developed. Such nonlinearity was treated with the sector condition. Optimization procedures, as $\mathcal{L}_2$ gain, maximization of the region of attraction and the maximum disturbance level such that the closed-loop system is ISS stable, were also presented. The conditions were extended to the case where an integrator over the tracking error is used, so the closed-loop system is capable of tracking piecewise constant references.

In Chapter 4, by means of sum of squares decomposition, conditions for synthesis of state-feedback controllers for LPV continuous-time systems with state polynomial dependency were developed. The smoothed approximation of the time-varying parameter was also used, hence the stabilization is provided even when it presents piecewise continuous behavior. The conditions were extended to compute controllers that minimize the $\mathcal{L}_2$ gain between the input disturbance and the system's output. It is important to notice that the approaches deal with time-varying parameters and state polynomial dependency simultaneously.

In Chapter 5, input constraints were considered when designing state-feedback controllers for continuous-time LPV systems with state polynomial dependency. As a design requirement, the $\mathcal{L}_2$ gain between the input disturbance and the system's output is considered. Similarly to Chapter 3, the conditions were extended to the case of piecewise constant reference tracking.

6.1 Developed Works

1. Published: State-feedback control for LPV polynomial continuous-time systems. This paper is concerned with the design of state-feedback controllers for LPV polynomial continuous-time systems, as presented in Chapter 4. Published in *21st* IFAC World Congress 2020.

2. Submitted: ISS control for continuous-time systems with filtered time-varying parameter and saturating actuators, as presented in Chapter 3.

3. In progress: Control of polynomial systems with input constraints, as presented in Chapter 5.

6.2 Future Works

Despite the fact the conditions developed presented good results, there are some topics that could be investigated, as:

1. For the cases of reference tracking and saturated actuator, estimate the maximum amplitude of reference input such that the closed-loop system presents null steady-state error and do not leave $\mathcal{R}_{\mathcal{A}}$.

2. For the synthesis conditions presented in Chapter 4, extend the approaches so the matrices $P(\xi)$ can also depend on the states, that is, $P(\xi,x)$.

3. For the polynomial systems in Chapter 5, apply the sector condition to deal with input constraints.

4. Develop the conditions proposed by using the well known Finsler's Lemma to compare with the results obtained.

References

Alam, S., & Ahmed, P. 2017. Several chaotic analysis of Lorenz system. *European scientific journal*, **13**(9), 438–455.

Barmish, B. R. 1985. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *Journal of optimization theory and applications*, **46**(4), 399–408.

Boyd, S., Ghaoui, L. El, Feron, E., & Balakrishnan, V. 1994. *Linear matrix inequalities in systems and control theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics.

Briat, C. 2015a. *Linear parameter-varying and time-delay systems — analysis, observation, filtering and control*. Advances in Delays and Dynamics, vol. 3. Berlin Heidelberg: Springer-Verlag.

Briat, C. 2015b. Stability analysis and control of a class of LPV systems with piecewise constant parameters. *Systems & control letters*, **82**(August), 10–17.

Briat, C. 2018. Stability analysis and state-feedback control of LPV systems with piecewise constant parameters subject to spontaneous poissonian jumps. *Ieee control systems letters*, **2**(4), 230–235.

Briat, C., & Mustafa, K. 2017. Stability analysis of LPV systems with piecewise differentiable parameters. *Ifac-papersonline*, **50**(07), 7554–7559.

Cherifi, A., Guelton, K., Arcese, L., & Leite, V. J. S. 2019. Global non-quadratic d-stabilization of Takagi–Sugeno systems with piecewise continuous membership functions. *Applied mathematics and computation*, **351**(1), 23–36.

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. 2007. Robust stability of time-varying polytopic systems via parameter-dependent homogeneous Lyapunov functions. *Automatica*, **43**(2), 309–316.

de Oliveira, J. 2000. *Controle robusto de sistemas lineares a parâmetros variantes no tempo: técnicas de análise e síntese.* Tech. rept. http://repositorio.ufsc.br/xmlui/handle/123456789/78472. UFSC.

de Souza, C., Leite, V. J. S., Silva, L. F. P., & Castelan, E. B. 2019. ISS robust stabilization of state-delayed discrete-time systems with bounded delay variation and saturating actuators. *Ieee transactions on automatic control*, **64**(9), 3913–3919.

Delibaş, A., Kucukdemiral, I. B., & Cansever, G. 2013. $\mathcal{L}_2$ control of LPV systems with saturating actuators: Pólya Approach. *International Journal of Robust and Nonlinear Control*, **34**, 17–34.

Ebenbauer, C., & Allgöwer, F. 2006. Analysis and design of polynomial control systems using dissipation inequalities and sum of squares. *Computers & chemical engineering*, **30**, 1590–1602.

Ebenbauer, C., Raff, T., & Allgower, F. 2005. A duality-based LPV approach to polynomial state feedback design. *Pages 703–708 of: Proceedings of the 2005 american control conference.*

Ferreira, G. O., Lacerda, M. J., & Leite, V. J. S. 2020. State-feedback control for continuous-time LPV systems with polynomial vector fields. *International Federation of Automatic Control.*

Figueiredo, L. S. 2020. *Controle de sistemas LPV discretos no tempo sujeitos a atuadores ssaturantes.* M.Sc. Thesis, CEFET-MG – Centro Federal de Educação Tecnológica de Minas Gerais, Divinópolis, MG.

Figueiredo, L. S., Parreiras, T. A. R., Lacerda, M. J., & Leite, V. J. S. 2020. Design of LPV-PI-like controller with guaranteed performance for discrete-time systems under saturating actuators. *IFAC-PapersOnline*, 1–6.

Geromel, J. C., & Colaneri, P. 2006. Robust stability of time varying polytopic systems. *Systems & control letters*, **55**(1), 81–85.

Gilbert, W., Henrion, D., Bernussou, J., & Boyer, D. 2010. Polynomial LPV synthesis applied to turbofan engines. *Control engineering practice*, **18**, 1077–1083.

Gomes da Silva Jr., J. M., & Tarbouriech, S. 2005. Antiwindup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, **50**(1), 106–111.

Groff, L. B., Gomes da Silva Jr., J. M., & Valmórbida, G. 2019a. Regional stability of discrete-time linear systems subject to asymmetric input saturation. *Pages 169–174 of: Proceedings of the ieee conference on decision and control.*

Groff, L. B., Valmórbida, G., & Gomes da Silva Jr., J. M. 2019b. Stability analysis of piecewise affine discrete-time systems. *Pages 8172–8177 of: Proceedings of the ieee conference on decision and control.*

Hanema, J., Lazar, M., & Tóth, R. 2020. Heterogeneously parameterized tube model predictive control for LPV systems. *Automatica*, **111**.

Horisberger, H. P., & Belanger, P. R. 1976. Regulators for linear, time invariant plants with uncertain parameters. *Ieee transactions on automatic control*, **21**, 705–708.

Hu, T., & Lin, Z. 2001. *Control systems with actuator saturation: Analysis and design.* 1 edn. Control Engineering. charlottesville, VA, USA: Birkhäuser Basel.

Jennawasin, T., & Banjerdpongchai, D. 2018. Design of state-feedback control for polynomial systems with quadratic performance criterion and control input constraints. *Systems & control letters*, **117**, 53–59.

Jungers, M., & Castelan, E. B. 2011. Gain-scheduled output control design for a class of discrete-time nonlinear systems with saturating actuators. *Systems & Control Letters*, **60**(3), 169–173.

Karimi, H. R., Moshiri, B., Maralani, P. J., & Lohmann, B. 2005. Adaptative $\mathcal{H}_\infty$- control design for a class of LPV systems. *44th ieee conference on decision and control and the european control conference*, 7918–7923.

Löfberg, J. 2004 (September). YALMIP: A toolbox for modeling and optimization in MATLAB. *Pages 284–289 of: Proceedings of the 2004 ieee international symposium on computer aided control systems design.*

Lopes, A. N. D., Leite, V. J. S., & Silva, L. F. P. 2018. On the integral action of discrete-time fuzzy TS control under saturated actuator. *IEEE international conference on fuzzy systems (FUZZ-IEEE)*, 1–8.

Montagner, V. F., & Peres, P. L. D. 2003 (December). A new LMI condition for the robust stability of linear time-varying systems. *Pages 6133–6138 of: Proceedings of the 42nd ieee conference on decision and control.*

Montagner, V. F., Oliveira, R. C. L. F., Leite, V. J. S., & Peres, P. L. D. 2005. LMI approach for $\mathcal{H}_\infty$ linear parameter-varying state-feedback control. *Iee proceedings - control theory and applications*, **152**(2), 195–201.

Montagner, V. F., Oliveira, R. C. L. F., Peres, P. L. D., Tarbouriech, S., & Queinnec, I. 2007. Gain-scheduled controllers for linear parameter-varying systems with saturating actuators: LMI-based design. *Pages 6067–6072 of: Proc. of the 2007 American Control Conference.*

Mozelli, L. A., & Adriano, R. L. S. 2016. On computational issues for stability analysis of LPV systems using parameter-dependent lyapunov functions and LMIs. *Journal of the franklin institute*, **353**, 781–796.

Márquez, R., Guerra, T. M., Bernal, M., & Kruszewski, A. 2016. A non-quadratic lyapunov functional for $\mathcal{H}_\infty$ control of nonlinear systems via takagi-sugeno models. *Journal of the franklin institute*, **353**(4), 781–796.

Nisha, Pal, V. C., Negi, R., & Pati, A. 2018. Stability analysis of continuous time-delayed system with input saturation. *5th ieee uttar pradesh section international conference on electrical, electronics and computer engineering*, 1–6.

Oliveira, R. C. L. F., & Peres, P. L. D. 2009. Time-varying discrete-time linear systems with bounded rates of variation: Stability analysis and control design. *Automatica*, **45**(11), 2620–2626.

Papachristodoulou, A., Anderson, J., Valmorbida, G., Prajna, S., Seiler, P., & Parrilo, P. A. 2013. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB.* `http://arxiv.org/abs/1310.4716`. Available from `http://www.eng.ox.ac.uk/control/sostools`.

Peixoto, M. L. C., Pessim, P. S. P., Lacerda, M. J., & Palhares, R. M. 2020. Stability and Stabilization for LPV systems based on Lyapunov functions with non-monotonic terms. *Journal of the franklin institute*, **357**(11), 6595–6614.

Prajna, S., Parrilo, P. A., & Rantzer, A. 2004. Nonlinear control synthesis by convex optimization. *Ieee transactions on automatic control*, **49**(2), 310–314.

Rakhshan, M., Vafamand, N., Mardani, M. M., Khooban, M. H., & Dragičević, T. 2018. Polynomial control design for polynomial systems: A non-iterative sum of squares approach. *Transactions of the institute of measurement and control*, **41**(7), 1993–2004.

Slotine, J. J. E., & Li, W. (eds). 1991. *Applied nonlinear control.* New Jersey: Prentice-Hall.

Sontag, E. D. 1999. *Nonlinear analysis, differential equations and control.* Vol. 528. Chap. Stability and stabilization: discontinuities and the effect of disturbances, pages 551–598.

Sturm, J. F. 1999. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, **11**(1–4), 625–653. `http://sedumi.ie.lehigh.edu/`.

Tanaka, K., Yoshida, H., Ohtake, H., & Wang, H. O. 2009. A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. *Ieee transactions on fuzzy systems*, **17**(4), 911–922.

Tarbouriech, S., Garcia, G., Gomes da Silva Jr., J. M., & Queinnec, I. 2011a. *Stability and stabilization of linear systems with saturating actuators.* Springer Science & Business Media.

Tarbouriech, S., Garcia, G., Gomes da Silva Jr., J. M., & Queinnec, I. 2011b. *Stability and stabilization of linear systems with saturating actuators.* Springer Science & Business Media.

Tarbouriech, S., Garcia, G., Gomes da Silva Jr., J. M., & Queinnec, I. 2011c. *Stability and stabilization of linear systems with saturating actuators.* London, UK: Springer.

Valmórbida, G., Tarbouriech, S., & Garcia, G. 2013. Design of polynomial control laws for polynomial systems subject to actuator saturation. *Ieee transactions on automatic control*, **58**(7), 1758–1770.

Wang, L., & Liu, X. 2011. Gain scheduled state feedback control for discrete-time-varying polytopic systems subject to input saturation. *Circuits, systems, and signal processing*, **30**, 1665–1182.

Wu, F., & Prajna, S. 2005. SOS-based solution approach to polynomial LPV system analysis and synthesis problems. *International journal of control*, **78**(8), 600–611.

Xiang, W. 2018. Parameter-memorized Lyapunov functions for discrete-time systems with time-varying parametric uncertainties. *Automatica*, **87**(January), 450–454.

Yamamoto, K., Sename, O., Koenig, D., & Moulaire, P. 2019. Design and experimentation of an LPV extended state feedback control on electric power steering systems. *Control engineering practice*, **90**, 123–132.

Yang, S., & Wu, F. 2014. Control of polynomial nonlinear systems using higher degree lyapunov functions. *Journal of dynamic systems measurement and control*, **136**, 031018–.

Zhang, J., Xu, X., Yang, L., & Yang, X. 2019. LPV model-based multivariable indirect adaptive control of damaged asymmetric aircraft. *Journal of aerospace engineering*, **32**(6), 04019095.

Zhao, D., & Wang, J. 2009. An improved nonlinear $\mathcal{H}_\infty$ synthesis for parameter-dependent polynomial nonlinear systems using sos programming. *Pages 796–801 of: 2009 american control conference*.

Zheng, Q., & Wu, F. 2009. Regional stabilisation of polynomial non-linear systems using rational lyapunov functions. *International journal of control*, **82**, 1605–1615.